SINGLE LINE IN SPACE SOLUTION TO GEOCENTRIC COORDINATES OF AN EARTH SURFACE POINT

Prepared by

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Approved For Release 2003/06/20 : CIA-RDP80T00703A000800010001-3

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1. INTRODUCTION

The usual method of determining the coordinates of a point on the earth's surface photogrammetrically is with intersection equations formed with the space camera coordinates and orientation matrices of an overlapping pair of photographs on which the object point is imaged. The usual method is the desirable method when the intersection angles or, sometimes referred to as parallax angles, are 10° or more. A single line solution requiring the space camera coordinates and the geocentric orientation matrix of the camera x,y,z axis provides a direct geometrically accurate solution when intersection lines enclose less than 10° angles. The single line solution is independent of camera cone angle geometry or the base-height ratios.

Space born panoramic photography has the advantage of high resolution paraxial imagery and wide cone angle normal to the direction of motion but a limitation in an extremely narrow cone angle in the direction of motion. This results in weak intersection angles along the orbital path.

2. GEOMETRY OF THE SINGLE LINE DETERMINATION

Assume an orbital point o in space whose coordinates have been determined with time T and known orbital parameters. Also assume that the direction cosines of a line to an unknown point on the earth's surface are known by some preprocessing means to be expanded later. This arrangement results in a line in space piercing an ellipsoid of revolution from a known point o in space. The intersection of the line in space with the ellipsoid of revolution defines the geocentric position of unknown point. The geometry of the solution is illustrated in Figure 1. The ellipsoid of revolution is expressed with the equation:

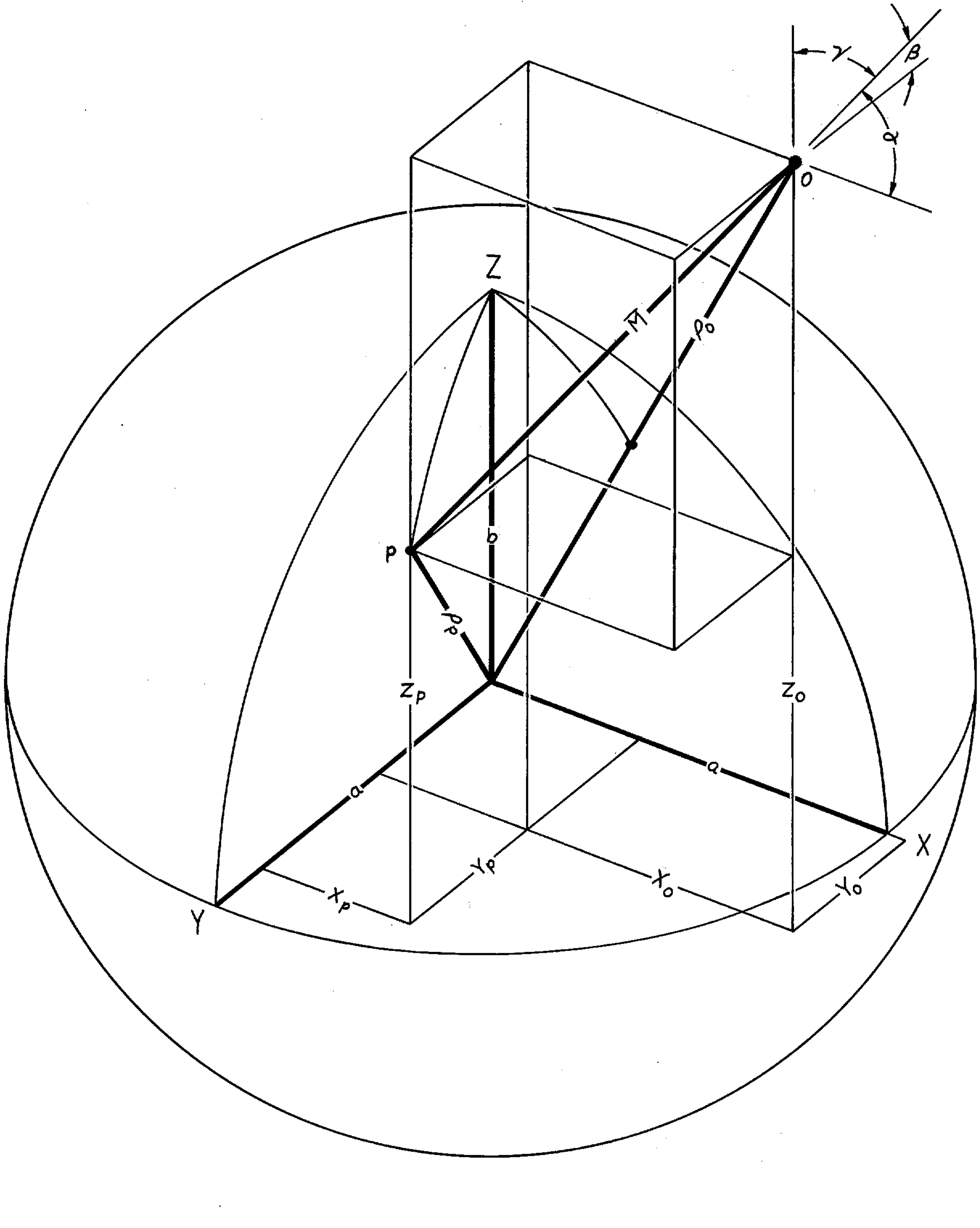

Figure 1 GEOMETRY OF SOLUTION

$$\frac{X^2 + Y^2}{a^2} + \frac{Z^2}{b^2} = 1$$

Where X, Y, Z are the geocentric coordinates of a surface point to be determined; a and b are the semi-major and minor axis of the earth. The length of the line in space, $\bar{M}$, may be expressed:

$$(X_o - X) \cos \alpha + (Y_o - Y) \cos \beta + (Z_o - Z) \cos \gamma = M$$

or

$$(X_o - X)^2 + (Y_o - Y)^2 + (Z_o - Z)^2 = M^2$$

also

$$X_o - X = M \cos \alpha$$
$$Y_o - Y = M \cos \beta$$
$$Z_o - Z = M \cos \gamma$$

or

$$X = X_o - M \cos \alpha$$
$$Y = Y_o - M \cos \beta$$
$$Z = Z_o - M \cos \gamma$$

Changing the form of the ellipsoid of revolution

$$(X^2 + Y^2) b^2 + Z^2 a^2 = a^2 b^2$$
$$b^2 = a^2 - c^2$$
$$e = \frac{c}{a}$$
$$a^2 e^2 = c^2$$
$$b^2 = a^2(1 - e^2)$$

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or

$$(X^2 + Y^2)\ a^2\ (1 - e^2) + Z^2 a^2 = a^4\ (1 - e^2)$$

Division by a^2

$$(X^2 + Y^2)\ (1 - e^2) + Z^2 = a^2\ (1 - e^2)$$

Substituting

$$[(X_o - M \cos \alpha)^2 + (Y_o - M \cos \beta)^2]\ (1 - e^2) + (Z_o - M \cos \gamma)^2$$

$$= a^2 (1 - e^2)$$

Expanding the squares

$$[M^2 (\cos^2 \alpha + \cos^2 \beta)\ (1 - e^2) - 2M\ (\cos \alpha X_o + \cos \beta Y_o)\ (1 - e^2)$$

$$+ M^2 \cos^2 \gamma - 2M \cos \gamma Z_o + [(X_o{}^2 + Y_o{}^2)\ (1 - e^2) + Z_o{}^2 - a^2 (1 - e^2) = 0$$

Collecting coefficients of descending powers of M

$$\begin{aligned} &M^2 [(\cos^2 \alpha + \cos^2 \beta)\ (1 - e^2) + \cos^2 \gamma)] \\ &- M\ 2\ [(\cos \alpha X_o + \cos \beta Y_o)\ (1 - e^2) + \cos \gamma Z_o] \\ &+ [(X_o{}^2 + Y_o{}^2)\ (1 - e^2) + Z_o{}^2 - a^2 (1 - e^2)] = 0 \end{aligned}$$

which is a quadratic in M

$$\begin{aligned} \text{Let } A &= [(\cos^2 \alpha + \cos^2 \beta)\ (1 - e^2) + \cos^2 \gamma)] \\ B &= -\ 2\ [(\cos \alpha X_o + \cos \beta Y_o)\ (1 - e^2) + \cos \gamma Z_o] \\ C &= [(X_o{}^2 + Y_o{}^2)\ (1 - e^2) + Z_o{}^2 - a^2 (1 - e^2)] \end{aligned}$$

Then

$$M = \frac{-B \pm \sqrt{B \quad - 4AC}}{2A}$$

whence

$$X = X_o - M \cos \alpha$$
$$Y = Y_o - M \cos \beta$$
$$Z = Z_o - M \cos \gamma$$

The quadratic yields two values for M. The obviously correct value is the shorter one inasmuch as the longer line is the continuation of the line after it pierces the first surface at the point of emergence. The two values of M are illustrated in Figure 2. The solution assumes that the elevation of the point is zero. The solution without modification is directly applicable to shore line points and low lying coastal regions at any latitude.

An elevated point will be displaced along M away from the camera station toward the earth's surface. A combination of a high elevation and large vertical angles enclosed between the line in space and the camera radius vector can cause large errors in both the length M and the geocentric coordinates (X, Y, Z of the point if the elevation h is neglected. The error is not caused by neglecting h in the reduction of latitude and longitude but by the displacement when the point is projected to the earth's surface.

Three rigorous solutions are described that yield the correct geocentric coordinates for a point within elevation h. A common property of each of the solutions is the solution for M. When $h \neq 0$ simple additional computations are performed to accurately correct for the displacement. Historically an approximate correction was made by differentiating the equation:

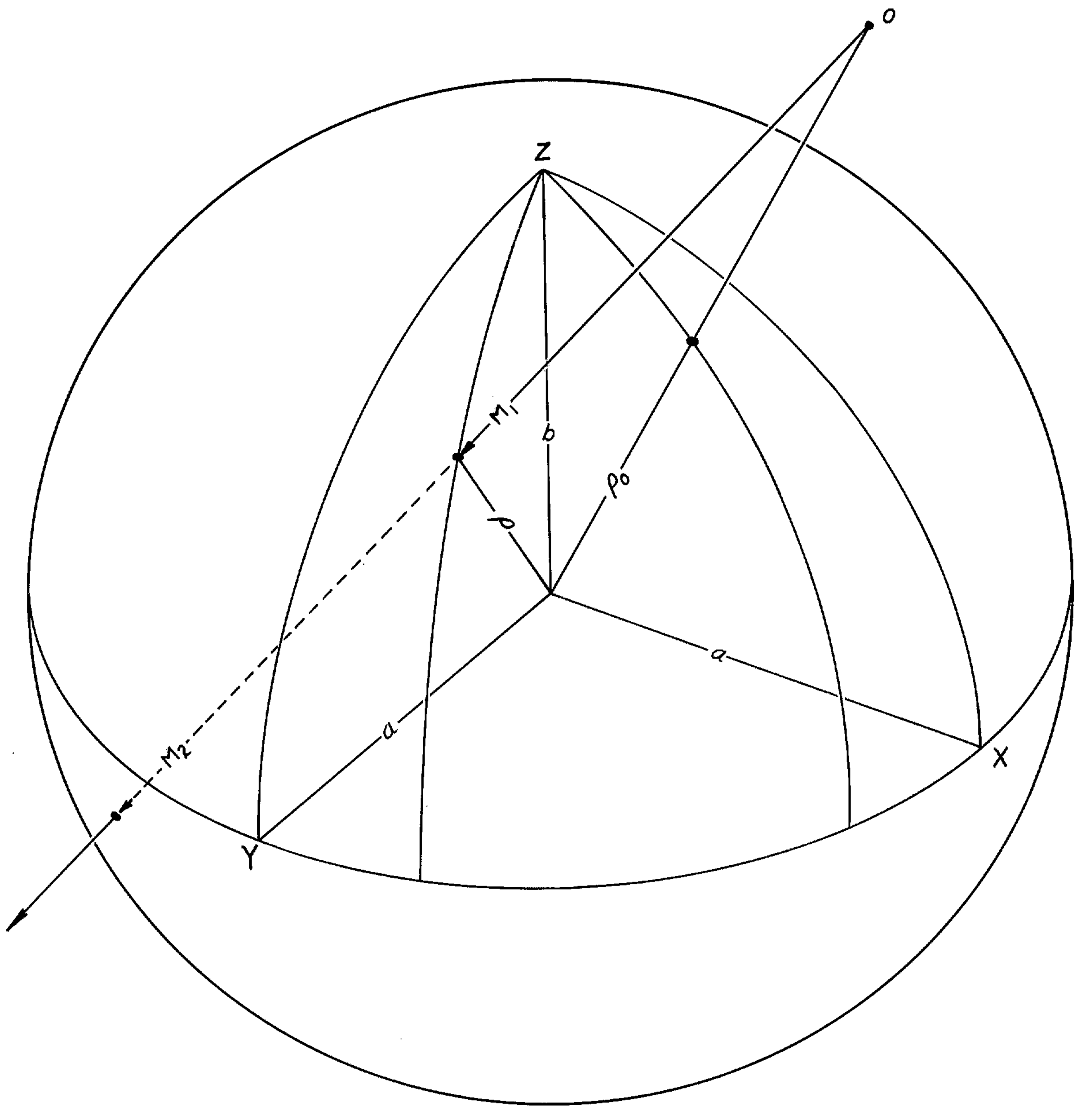

Figure 2 GEOMETRY OF TWO VALUES OF M

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$$\rho^2 = M^2 + \rho_o^{\ 2} - 2M\rho_o \cos\zeta_o$$

where

$$\rho = (X^2 + Y^2 + Z^2)^{\frac{1}{2}}$$

$$\rho_o = (X_o^{\ 2} + Y_o^{\ 2} + Z_o^{\ 2})^{\frac{1}{2}}$$

$$\cos\zeta_o = \cos\alpha\,\frac{X_o}{\rho_o} + \cos\beta\,\frac{Y_o}{\rho_o} + \cos\gamma\,\frac{Z_o}{\rho_o}$$

Differentiating M with respect to ρ

$$2\rho\,.\,d\rho = 2M\,.dM - 2\rho_o \cos\zeta_o dM$$

n $$dM = \frac{\rho\,.\,d\rho}{M - \rho_o \cos\zeta_o}$$

putting $d\rho = h$

$$dM = \frac{\rho\,.\,h}{M - \rho_o \cos\zeta_o}$$

The geometric significance of this equation is shown in Figure 3. The ratio $\dfrac{\rho}{M - \rho_o \cos\zeta_o} = \sec\zeta$ and $\sec\zeta\,.\,h = dM$

Because the equation neglects earth curvature dM is over corrected placing P at P_1. In any case, assuming dM to be correct

$$X_P = X - dM\cos\alpha$$
$$Y_P = Y - dM\cos\beta$$
$$Z_P = Z - dM\cos\gamma$$

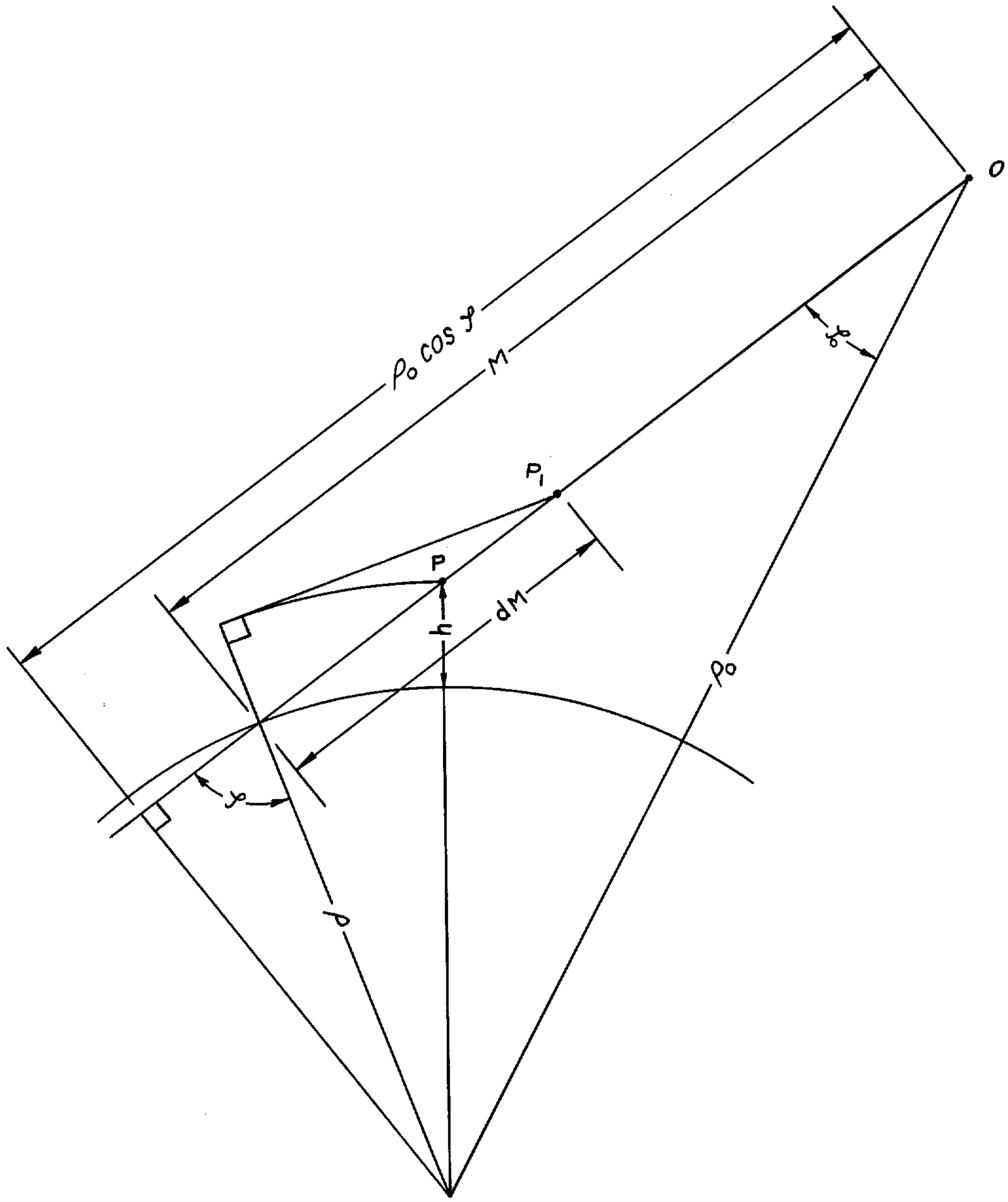

Figure 3 APPROXIMATE ADJUSTMENT TO M FOR h

The three methods described remove the errors arising from the above approximations.

For completeness for the case where h = 0 the geodetic latitude ϕ and longitude λ are as follows:

$$\tan \phi = \frac{Z}{(X^2 + Y^2)^{\frac{1}{2}} \quad (1 - e^2)}$$

$$\tan \lambda = \frac{Y}{X}$$

and the geocentric latitude ϕ' is simply $\tan \phi' = \frac{Z}{(X^2 + Y^2)^{\frac{1}{2}}}$

also for other applications

$$\sin \phi = \frac{Z}{[(X^2 + Y^2)(1 - e^2) + Z^2]^{\frac{1}{2}}}$$

$$\cos \phi = \frac{(X^2 + Y^2)(1 - e^2)}{[(X^2 + Y^2)(1 - e^2) + Z^2]^{\frac{1}{2}}}$$

$$\sin \lambda = \frac{Y}{(X^2 + Y^2)^{\frac{1}{2}}}$$

$$\cos \lambda = \frac{X}{(X^2 + Y^2)^{\frac{1}{2}}}$$

There is no justification for neglecting h. There are very few land platforms in the world that the elevation can not be interpolated from available maps to several hundred feet.

It was assumed in developing the quadratic for M that the direction cosines of the line $\cos \alpha$, $\cos \beta$, $\cos \gamma$ and the geocentric coordinates of o, X_o, Y_o, Z_o were known. The determination of these values is considered preliminary data processing.

The next section covers the various forms in which the orientation and the spatial camera coordinates may have and the procedure for transforming the various given forms to the necessary geocentric reference orientation of the camera axis and coordinates of the camera station.

3. REDUCTION OF GEOCENTRIC CAMERA ORIENTATION AND COORDINATES

3.1 Geocentric Coordinates of the Camera Station

The advantage of geocentric coordinates is the freedom from problems of earth curvature and local tangent planes changing rapidly with each position of the camera in orbit.

The geocentric coordinates of the camera station are computed with Keplerian equations of an orbit. While these equations do not account for periodic perturbations modern orbital computing programs output orbital parameters at sufficiently close intervals to justify use of the simpler Keplerian equations. An earth centered orbiting satellite is illustrated in Figure 4.

(1) X_s, Y_s, Z_s denote an earth centered star-referenced coordinate system referred to the vernal equinox in the plane of the earth's equator.

(2) X_o, Y_o, Z_o are geocentric coordinates of a point or camera station on the orbital path at clock time T.

(3) Perigee and apogee are near and far points of the orbital path. These points define the long axis ($2\ a_o$) of the orbit.

(4) An elliptical orbit has an eccentricity e_o defined by

$$e_o = \frac{c_o}{a_o}$$

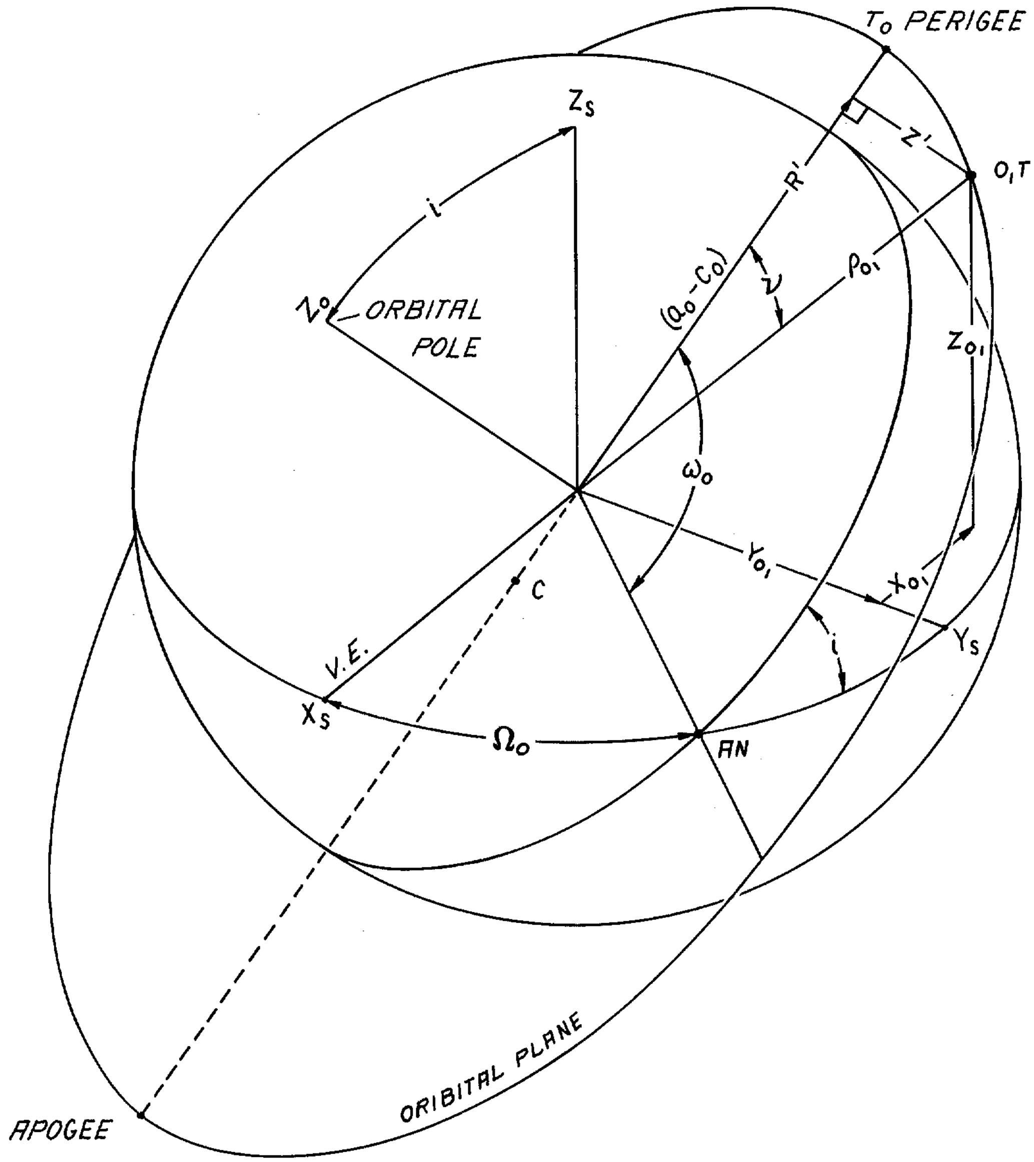

Figure 4a ORBITAL PARAMETERS

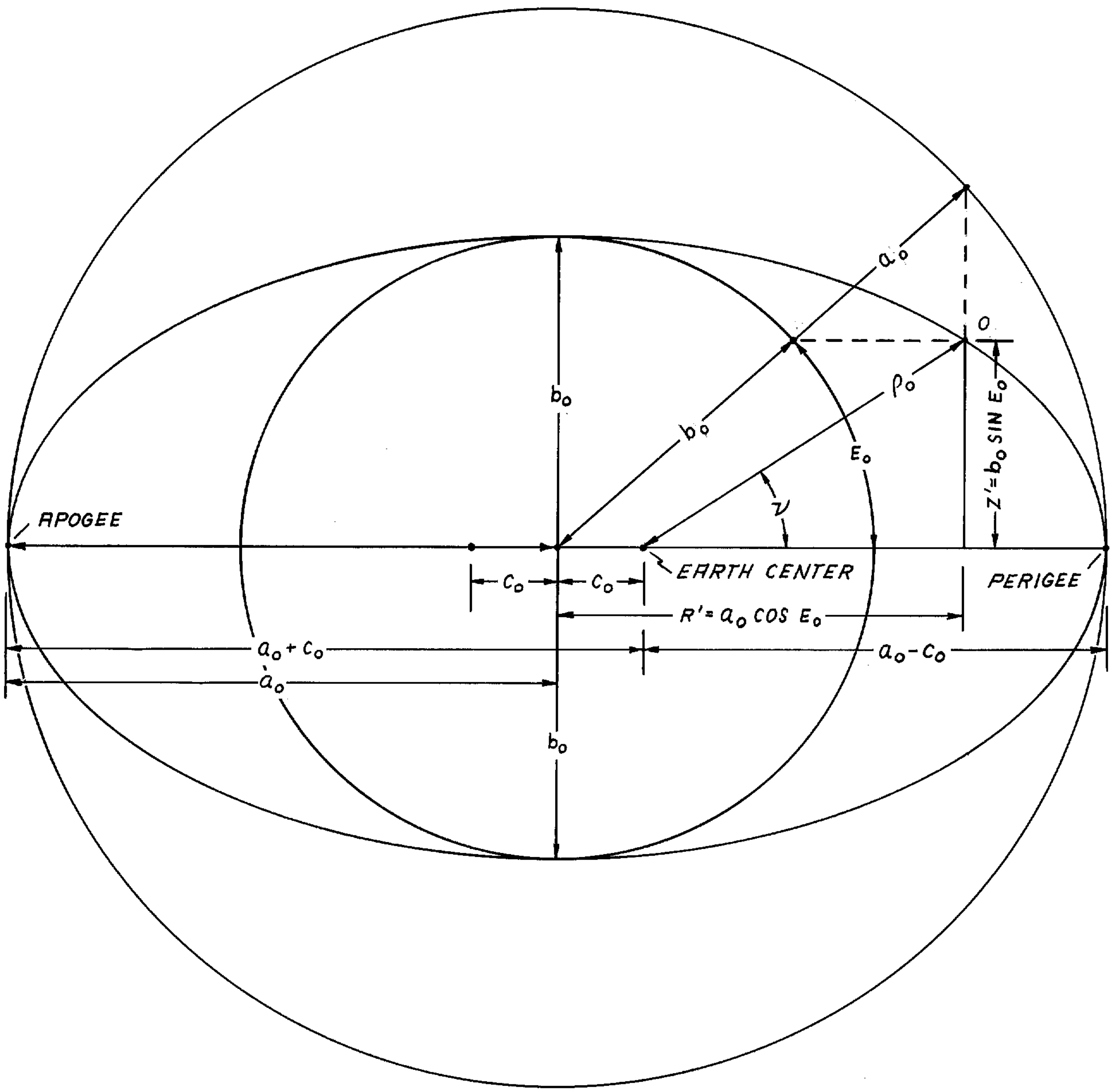

Figure 4b RELATION OF CENTER OF EARTH TO CENTER OF ORBIT AND PARAMETRIC ANGLES E

c_o is the distance along the long axis between the geometric center of the orbit and the center of the earth.

(5) $a_o - c_o$ is the distance from perigee to the center of the earth.

(6) $a_o + c_o$ is the distance from apogee to the center of the earth.

(7) b_o is the short diameter of the orbit at the geometric center and perpendicular to the long axis.

(8) The path of the satellite defines a plane neglecting perturbations that is space oriented just as the vernal equinox is space oriented (fixed in space).

(9) The intersection points of the plane of the orbit with the plane of the celestial equator are called nodes. The ascending node is the part where the satellite moves northward whereas the descending node is the part where the satellite moves southward.

(10) ι is the inclination of the plane of the orbit to the plane of the equator.

(11) Ω is the right ascension of the ascending node or the angle measured eastward from the vernal equinox in the plane of the celestial equator to the ascending node.

(12) ω is the angle subtended by the ascending node and perigee at the center of the earth.

(13) ν is the angle subtended by the camera at any time T and perigee point at time T_o.

(14) E_o is a parametric angle that defines the coordinates of a point referred to the orbital a_o and b_o axes.

$$R'_o = a_o \cos E_o$$

$$Z'_o = b_o \sin E_o$$

(15) n is the mean radian velocity of an orbiting particle.

(16) GM is the product of a gravitational constant and the mass of the earth.

(17) P is the clock time of two successive perigee passages of a satellite.

Given orbital parameters a_o, e_o, Ω_o, ω_o, T_o and time T determine X_o, Y_o, and Z_o of the camera station referred to the vernal equinox:

$$n = \sqrt{\frac{GM}{a_o^3}}$$

where GM = $14076.92056 \times 10^{12}$ ft $^3/\text{sec}^2$. Parametric angle E_o is computed first by an iterative process:

$$n\ (T-T_o) = E_o - e_o \sin E_o$$

$$n\ (T-T_o) = E_o' \text{ (approximate } E_o)$$

$$n\ (T-T_o) - (E_o' - e_o \sin E_o') = ndT$$

$$\Delta E = \frac{ndt}{1 - e_o \cos E_o'}$$

$$E_o = E_o' + \Delta E$$

Seldom are more than two iterations required. The radius vector ρ_o follows

$$\rho_o = a_o\ (1 - e_o \cos E_o)$$

and then

$$\cos \nu_o = \frac{\cos E_o - e_o}{1 - e_o \cos E_o}$$

$$\sin \nu_o = \frac{(1 - e_o^2)1/2 \sin E_o}{1 - e_o \cos E_o}$$

$$\rho_o = \frac{a_o (1 - e_o^2)}{1 + e_o \cos \quad)}$$

also

$$\sin E_o + \frac{(1 - e_o^2) \sin \nu}{1 + e_o \cos \nu}$$

$$\cos E_o = \frac{e_o + \cos \nu}{1 + e_o \cos \nu}$$

The geocentric coordinates follow:

with law of cosines

$$\cos \alpha_o = \cos (\nu+\omega) \cos \Omega - \sin (\nu+\omega) \sin \Omega \cos \iota.$$
$$\cos \beta_o = \cos (\nu+\omega) \sin \Omega + \sin (\nu+\omega) \cos \Omega \cos \iota.$$
$$\cos \gamma_o = \sin (\nu+\omega) \sin \iota$$

whence

$$X_s = \rho_o \cos \alpha_o$$

$$Y_s = \rho_o \cos \beta_o$$

$$Z_s = \rho_o \cos \gamma_o$$

Since the earth is rotating it may be desirable to refer the X,Y coordinates to the Greenwich Meridian in place of the vernal equinox. This is accomplished by replacing Ω with λ

λ_o = GST arc − Ω

GST = GCT + sol − sid + GST o^h

GCT is the T of the exposure. In any case

$X_o = \rho_o[\cos(\nu + \omega)\cos\lambda_o - \sin(\nu + \omega)\sin\lambda_o\cos i$

$Y_o = \rho_o[\cos(\nu + \omega)\sin\lambda_o + \sin(\nu + \omega)\cos\lambda_o\cos i$

$Z_o = Z_s$ since Z_s is the axis of rotation.

3.2 Camera Orientation Matrix

It is most practical to refer the camera x, y, z axes to the earth geocentric X, Y, Z axes. In practice the camera orientation is either referred to the local vertical and local meridian or the sidereal geocentric coordinates depending on how the camera orientation is determined. If the camera orientation is determined with Greenwich reference geocentric object space coordinates with well-known collinearity equations the matrix has the form:

	x	y	z
X_g	$\cos\alpha_x$	$\cos\alpha_y$	$\cos\alpha_z$
Y_g	$\cos\beta_x$	$\cos\beta_y$	$\cos\beta_z$
Z_g	$\cos\gamma_x$	$\cos\gamma_y$	$\cos\gamma_z$

where

$$\cos\alpha_x = \sin\lambda_p\cos A_z + \cos\lambda_p\sin A_z\sin\delta_p$$
$$\cos\alpha_y = \sin\lambda_p\sin A_z - \cos\lambda_p\cos A_z\sin\delta_p$$
$$\cos\alpha_z = \cos\lambda_p\cos\delta_p$$

$$\cos \beta_x = - \cos \lambda_p \cos A_z + \sin \lambda_p \sin A_z \sin \delta_p$$

$$\cos \beta_y = -\cos \lambda_p \sin A_z - \sin \lambda_p \cos A_z \sin \delta_p$$

$$\cos \beta_z = \sin \lambda_p \cos \delta_p$$

$$\cos \gamma_x = \sin A_z \cos \delta_p$$

$$\cos \gamma_y = \cos A_z \cos \delta_p$$

$$\cos \gamma_z = \sin \delta_p$$

Where A_z is the angle between the camera y axis and the apparent local meridian λ_p is the apparent longitude of the optical axis translated parallel to itself to the center of the earth and δ_p is simply the angle the optical axis defines with the plane of the equator. If this is the form of the orientation matrix λ_o replaces Ω_o in the reduction of space coordinate so that both orientation and coordinates refers Greenwich.

Suppose the camera orientation is determined with star data exposed simultaneously with the terrain camera and having a known relative orientation with the star camera. This is the more likely form:

	x	y	z
X_s	$\cos \alpha x_s$	$\cos \alpha y_s$	$\cos \alpha z_s$
Y_s	$\cos \beta x_s$	$\cos \beta y_s$	$\cos \beta z_s$
Z_s	$\cos \gamma x_s$	$\cos \gamma y_s$	$\cos \gamma z_s$

where $\cos \alpha x_s = - \sin \Omega_p \cos A_z + \cos \Omega_p \sin A_z \sin \delta_p$

$\cos \alpha y_s = - \sin \Omega_p \sin A_z - \cos \Omega_p \cos A_z \sin \delta_p$

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$$\cos \alpha z = \cos \Omega_p \cos \delta_p$$

$$\cos \beta x_s = \cos \Omega_p \cos A_z + \sin \Omega_p \sin A_z \sin \delta_p$$

$$\cos \beta y_s = \cos \Omega_p \sin A_z - \sin \Omega_p \cos A_z \sin \delta_p$$

$$\cos \beta z_s = \sin \Omega_p \cos \delta_p$$

$$\cos \gamma x_s = \cos A_z \cos \delta_p$$

$$\cos \gamma y_s = \sin A_z \cos \delta_p$$

$$\cos \gamma z_s = \sin \delta_p$$

A_z and δ_p have the same meaning as before and Ω_p is the right ascension of the optical axis.

In still other cases the orientation matrix may be referred to the local vertical and the local tangent plane at the foot of the local vertical with object space Y lying in the local meridian. This form employed usually for altitudes under 10 miles has the form.

	x	y	z
East (X)	$\cos \alpha x$	$\cos \alpha y$	$\cos \alpha z$
North (Y)	$\cos \beta x$	$\cos \beta y$	$\cos \beta z$
Zenith (Z)	$\cos \gamma x$	$\cos \gamma y$	$\cos \gamma z$

where

$$\cos \alpha x = \cos S \cos A_z + \sin S \sin A_z \cos t$$
$$\cos \alpha y = \sin S \cos A_z - \cos S \sin A_z \cos t$$
$$\cos \alpha z = \sin A_z \sin t$$
$$\cos \beta x = -\cos S \sin A_z + \sin S \cos A_z \cos t$$

$$\cos \beta y = - \sin S \sin A_z - \cos S \cos A_z \cos t$$
$$\cos \beta z = \cos A_z \sin t$$
$$\cos \gamma x = \sin S \sin t$$
$$\cos \gamma y = \cos S \sin t$$
$$\cos \gamma z = \cos t$$

S, is a rotation about the optical axis from the camera y axis to the image of the direction of gravity. t, is the angle enclosed between the optical axis and the vertical and A_z is the angle in the horizontal plane at the foot of the vertical measured clockwise from the vertical projection of the optical axis to the Y axis lying in the local meridian.

The set of Eulerian angles defining each matrix is illustrated in Figures 5a, 5b.

The basic point of this section is that the orientation matrix must be geocentric whether it is referenced to the vernal equinox or the Greenwich Meridian. If the orientation matrix is geocentric referred to the vernal equinox the geocentric coordinates of the camera station are also referred to the vernal equinox. Only the end geocentric coordinates of the earth surface point need be rotated to be referenced to Greenwich or in the case of a star-referenced determination.

$$X_g - X_s \cos GST - Y_s \sin GST$$

$$Y_g = X_s \sin GST + Y_s \cos GST$$

$$Z_g = Z_s$$

If the orientation matrix is geocentric referenced to Greenwich then either GST − Ω replaces Ω in the reduction of

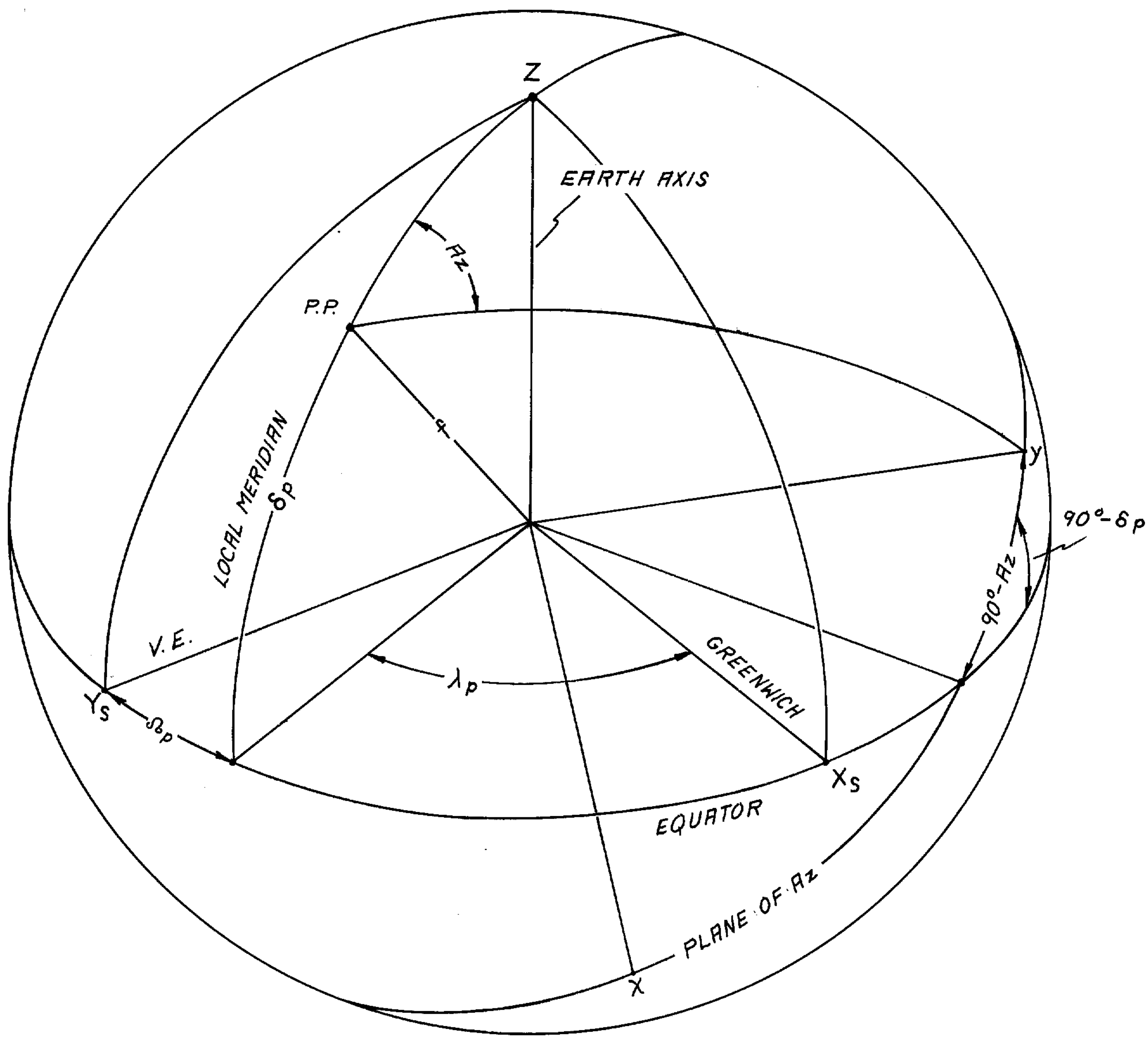

Figure 5a GEOCENTRIC ORIENTATION REFERENCED TO GREENWICH MERIDIAN OR THE VERNAL EQUINOX

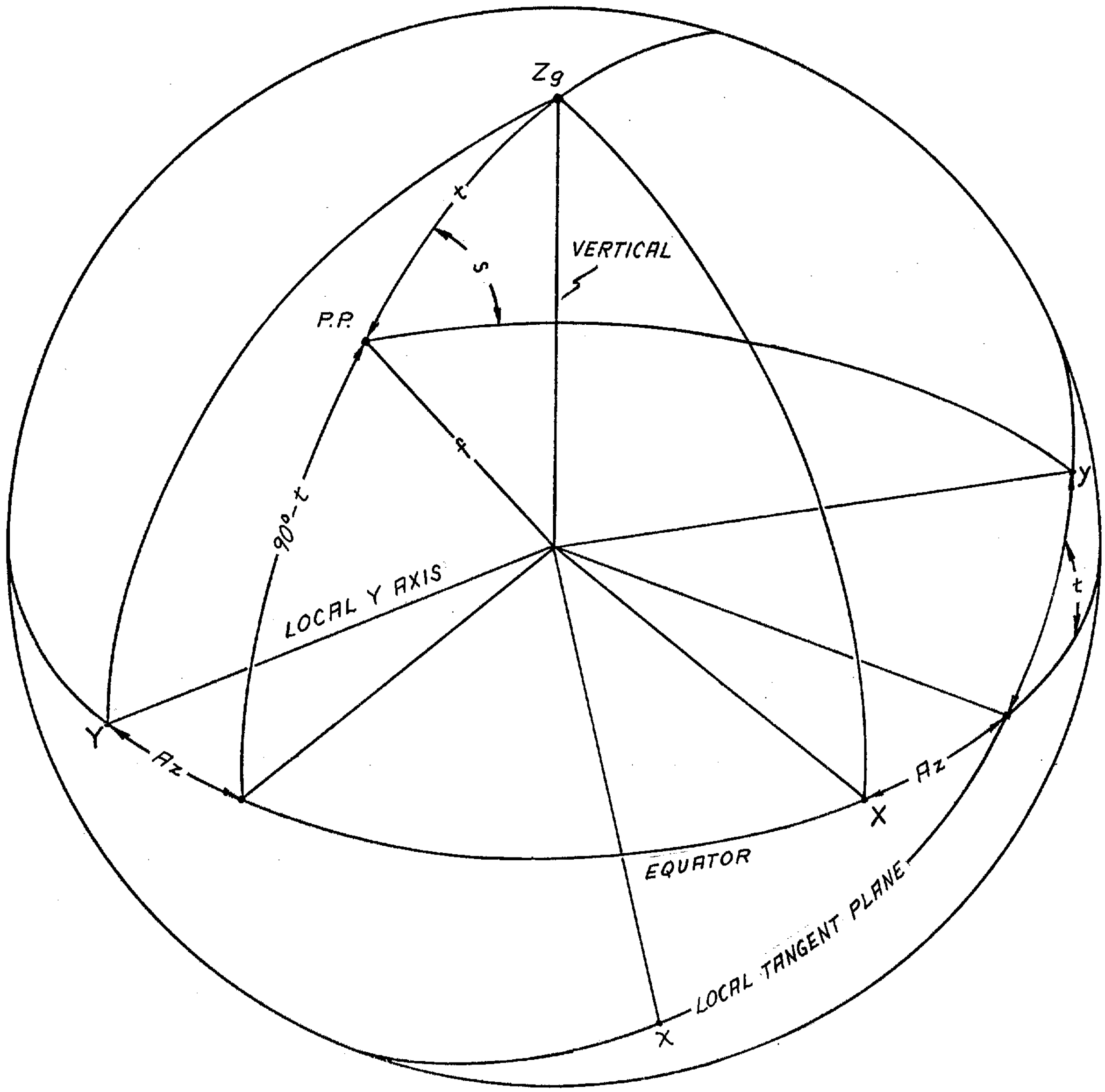

Figure 5b ORIENTATION REFERENCED TO LOCAL VERTICAL

camera station geocentric direction cosines or the vernal equinox geocentric coordinates of the camera station are rotated to Greenwich:

$$X_{o_g} = X_{o_s} \cos GST - Y_{o_s} \sin GST$$

$$Y_{o_g} = X_{o_s} \sin GST + Y_{o_s} \cos GST$$

$$Z_{o_g} = Z_{o_s}$$

Since the most universal form of the orientation matrix is star-referenced, let it be assumed that the geocentric coordinates of the camera will always be geocentric referred to the vernal equinox. Let it be assumed, further, if the orientation matrix is not star-referenced that it will be rotated to a star reference whether the original matrix refers to the local vertical or Greenwich. Consider first a rotation from the local vertical to a star reference. Given the orientation matrix referred to the local vertical and the sidereal Eulerian angles of the local vertical rotate the matrix to a sidereal reference.

	x	y	z
East (X)	$\cos \alpha x$	$\cos \alpha y$	$\cos \alpha z$
North (Y)	$\cos \beta x$	$\cos \beta y$	$\cos \beta z$
Zenith (Z)	$\cos \gamma x$	$\cos \gamma y$	$\cos \gamma z$

	X_s	Y_s	Z_s
East (X)	$\sin \Omega$	$\cos \Omega \sin \phi$	$\cos \Omega \cos \phi$
North (Y)	$\cos \Omega$	$\sin \Omega \sin \phi$	$\sin \Omega \cos \phi$
Zenith (Z)	0	$\cos \phi$	$\sin \phi$

where Ω is the right ascension of the local meridian and ϕ is the geodetic latitude defined on the local vertical.

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$$\cos \alpha x_s = \cos \alpha x \sin \Omega + \cos \beta x \cos \Omega$$
$$\cos \alpha y_s = \cos \alpha y \sin \Omega + \cos \beta y \cos \Omega$$
$$\cos \alpha z_s = \cos \alpha z \sin \Omega + \cos \beta z \cos \Omega$$
$$\cos \beta x_s = \cos \alpha x \cos \Omega \sin \phi + \cos \beta z \sin \Omega \sin \phi + \cos \gamma z \cos \phi$$
$$\cos \beta y_s = \cos \alpha y \cos \Omega \sin \phi + \cos \beta z \sin \Omega \sin \phi + \cos \gamma z \cos \phi$$
$$\cos \beta z_s = \cos \alpha z \cos \Omega \sin \phi + \cos \beta z \sin \Omega \sin \phi + \cos \gamma z \cos \phi$$
$$\cos \gamma x_s = \cos \alpha z \cos \Omega \cos \phi + \cos \beta x \sin \Omega \cos \phi + \cos \gamma x \sin \phi$$
$$\cos \gamma y_s = \cos \alpha y \cos \Omega \cos \phi + \cos \beta y \sin \Omega \cos \phi + \cos \gamma y \sin \phi$$
$$\cos \gamma z_s = \cos \alpha z \cos \Omega \cos \phi + \cos \beta z \sin \Omega \cos \phi + \cos \gamma z \sin \phi$$

Suppose it is desired to rotate from Greenwich to sidereal.

	x	y	z
X_g	$\cos \alpha x_g$	$\cos \alpha y_g$	$\cos \alpha z_g$
Y_g	$\cos \beta x_g$	$\cos \beta y_g$	$\cos \beta z_g$
Z_g	$\cos \gamma x_g$	$\cos \gamma y_g$	$\cos \gamma z_g$

	X_s	Y_s	Z_s
X_g	cos GST	sin GST	0
Y_g	- sin GST	cos GST	0
Z_g	0	0	1

$$\cos \alpha x_s = \cos \alpha x_g \cos GST - \cos \beta x_g \sin GST$$

$$\cos \alpha y_s = \cos \alpha y_g \cos GST - \cos \beta y_g \sin GST$$

$$\cos \alpha z_s = \cos \alpha z_g \cos GST - \cos \beta z_g \sin GST$$

$$\cos \beta x_s = \cos \alpha x_g \sin GST + \cos \beta x_g \cos GST$$

$$\cos \beta y_s = \cos \alpha y_g \sin GST + \cos \beta y_g \cos GST$$

$$\cos \beta z_s = \cos \alpha z_g \sin GST + \cos \beta z_g \cos GST$$

$$\cos \gamma x_s = \cos \gamma x_g$$

$$\cos \gamma y_s = \cos \gamma y_g$$

$$\cos \gamma z_s = \cos \gamma z_g$$

Clearly if a matrix can be rotated in one direction it can be rotated in the opposite or continue in the same direction (360°-GST) from sidereal to Greenwich or to the local vertical.

Henceforth, it will be assumed that the orientation matrix and the coordinates of the camera station are geocentric siderial referenced. Then given the camera coordinates of an image of an earth surface point the geocentric sidereal reference direction cosines of the line are computed:

$$\cos \alpha x_s \frac{x_i}{l_i} + \cos \alpha y_s \frac{y_i}{l_i} + \cos \alpha z_s \frac{f}{l_i} = \cos \alpha$$

$$\cos \beta x_s \frac{x_i}{l_i} + \cos \beta y_s \frac{y_i}{l_i} + \cos \beta z_s \frac{f}{l_i} = \cos \beta$$

$$\cos \gamma x_s \frac{x_i}{l_i} + \cos \gamma y_s \frac{y_i}{l_i} + \cos \gamma z_s \frac{f}{l_i} = \cos \gamma$$

where $\ell_i = (x_i^2 + y_i^2 + f^2)^{\frac{1}{2}}$

$\cos\alpha$, $\cos\beta$, $\cos\gamma$ of the line, X_o, Y_o, Z_o of the camera station, a, b of the earth and h the elevation of the point are the given data in the determination of the geocentric coordinates of the earth surface point.

4. THREE METHODS OF DETERMINING THE GEOCENTRIC COORDINATES OF AN EARTH SURFACE POINT IN WHICH $h \neq o$

4.1 Expanded Ellipsoid of Revolution Method

The geometry of the Expanded Ellipsoid Method is illustrated in Figure 6. The equatorial and polar radius of the earth are expanded to

$$a' = a + h$$

and

$$b' = b + h$$

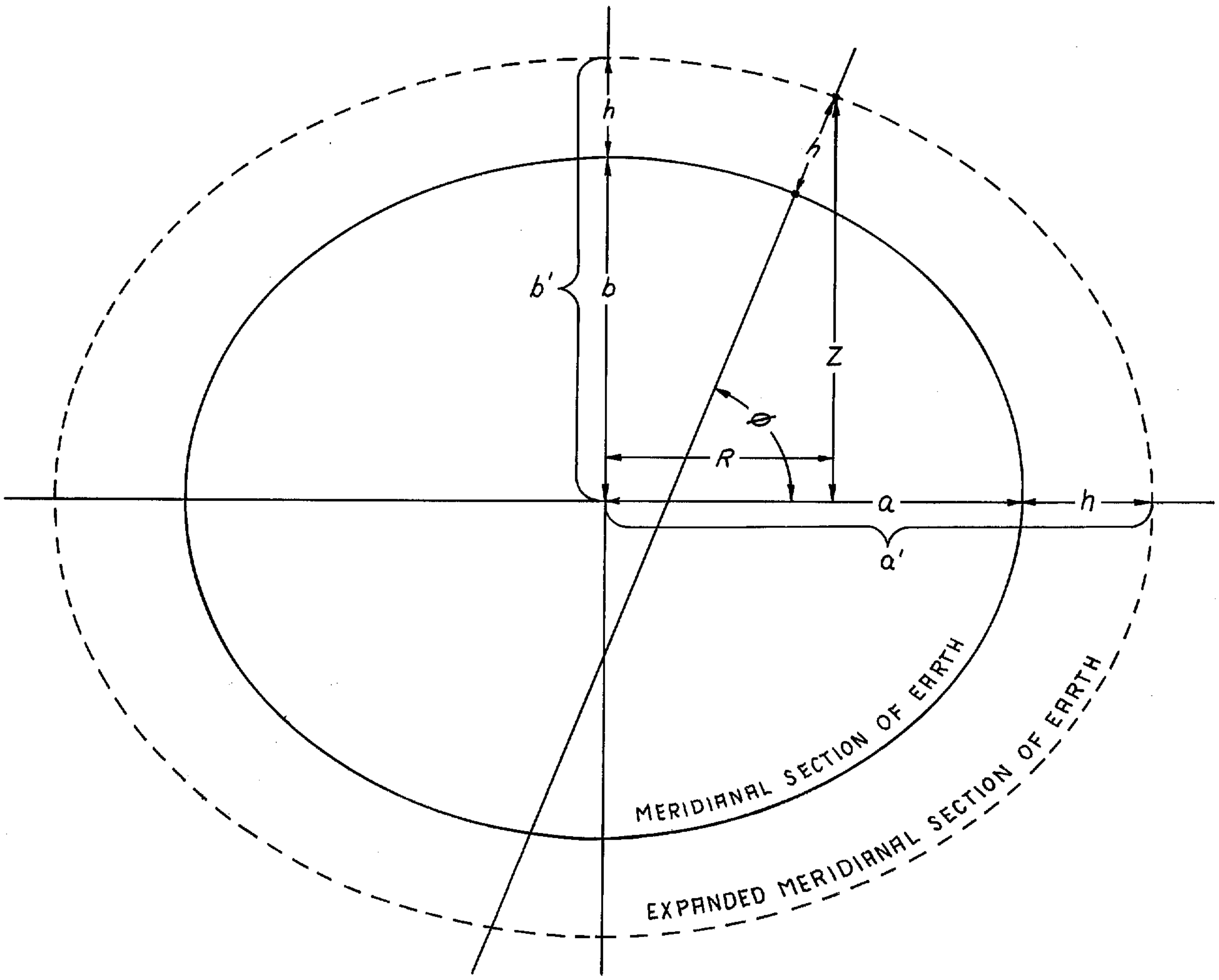
producing an enlarged ellipsoid of revolution that places the elevated point coincident with an imaginary datum. This removes the largest error arising from relief displacement in the quadratic determination M.

In the formation of the quadratic

$$e'^2 = \frac{(a + h)^2 - (b + h)^2}{(a + h)^2}$$

whence

$$A = [(\cos^2\alpha + \cos^2\beta)(1 - e'^2) + \cos^2\gamma]$$
$$B = -2\,[(X_o\cos\alpha + Y_o\cos\beta)(1 - e'^2) + Z_o\cos\gamma]$$



$$R = a \cos E + \frac{h \cos E (1-e^2)^{1/2}}{(1-e^2 \cos^2 E)^{1/2}}$$

$$Z = b \sin E + \frac{h \sin E}{(1-e^2 \cos^2 E)^{1/2}}$$

Figure 6 ELLIPSOID a AND b EXPANDED BY h

$$C = (X_o^2 + Y_o^2)(1 - e'^2) + Z_o^2 - a'^2(1 - e'^2)$$

and

$$M = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$X = X_o - M \cos \alpha$$
$$Y = Y_o - M \cos \beta$$
$$Z = Z_o - M \cos \gamma$$

Having employed a' and b' to determine the coordinates of a point on an expanded ellipsoid of revolution the accepted values of a and b are employed to determine latitude (ϕ). Longitude (λ) is independent of elevation and may be calculated directly:

$$\tan \lambda = \frac{Y}{X}$$

$$\sin \lambda = \frac{Y}{R}$$

$$\cos \lambda = \frac{X}{R}$$

where $R = (X^2 + Y^2)^{\frac{1}{2}}$

ϕ is determined directly or indirectly by an iterative process. In these reductions, iteration is employed as a computational convenience to avoid the less efficient explicit solution of a fourth degree polynomial.

Parametric angle is E is employed in the indirect method.

$$\tan E' = \frac{Z}{R}\ \frac{a}{b}$$

where

$$R = (X^2 + Y^2)^{\frac{1}{2}}$$

Then

$$\Delta = e^2 - \left[\frac{R}{a} \sec E' - \frac{Z}{a}\frac{b}{a} \csc E'\right]$$

$$\Delta E = \frac{\Delta}{\frac{R}{a} \sec E' \tan E' + \frac{Z}{a}\frac{b}{a} \csc E' \cot E'}$$

On the nth iteration

$$E = E' + \sum \Delta E$$

Where n is the iteration for which

$$\Delta = 0$$

With E determined

$$\tan \phi = \tan E \frac{a}{b} = \frac{\tan E}{(1 - e^2)^{\frac{1}{2}}}$$

In the direct method

$$\tan \phi' = \frac{Z}{R}\frac{a^2}{b^2}$$

$$\Delta = -[R \sec \phi'(1 - e^2) - Z \csc \phi' + h\, e^2]$$

$$\Delta\phi = \frac{\Delta}{R \sec \phi'(1 - e^2) \tan \phi' + Z \csc \phi' \quad \phi'}$$

and on the nth iteration

$$\phi = \phi' + \sum \Delta\phi$$

4.1.2 <u>Numerical Example</u>

Given

$$X_o = 8.314,238.0$$

$Y_o = 13,305,562.0$

$Z_o = 14,584,184.0$

$\cos \alpha = .87122760$

$\cos \beta = .46326245$

$\cos \gamma = -.16233007$

The camera station has an altitude of 100 miles at

$\phi_o = 43° \; 5' \; 45''.5$

$\lambda_o = 58° \; 0' \; 0''.0$

The true value of M, ϕ, λ, and h to be determined are as follows:

$M = 1,047,212.0$ ft.

$\phi = 45° \; 5' \; 46''.3$

$\lambda = 60°$

$h = 10,000.0$ ft.

Initially M is determined

$a' = a + h = 20,935,689.0$
$b' = b + h = 20,865,539.0$

$$e'^2 = \frac{a'^2 - b'^2}{2} = .00669025$$

$$1 - e'^2 = .99330975$$

$$A = [(\cos^2\alpha + \cos^2\beta)(1 - e'^2) + \cos^2\gamma]$$

$$= .99348672$$

$$B = -2[(\cos\alpha X_o + \cos\beta Y_o)(1 - e'^2) + \cos\gamma Z_o]$$

$$= -21,900,818.0$$

$$C = (X_o^2 + Y_o^2)(1 - e'^2) + Z_o^2 - a'^2(1 - e'^2)$$

$$= 21.845,337 \times 10^{12}$$

$$M = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

M = 1,047,215.0 which is only 3 ft. in error.

whence

	Error
$X = X_o - M\cos\alpha = 7,401,878.0$	0.0
$Y = Y_o - M\cos\beta = 12,820,428.0$	0.0
$Z = Z_o - M\cos\gamma = 14,754,178.0$	0.0

Thus, there is no error in the geocentric coordinates of the unknown point from the approximation of replacing a and b with a' and b'. Next the latitude, longitude are determined and the elevation is checked.

By the indirect method

$$R = (X^2 + Y^2)^{\frac{1}{2}} = 14,803,755.0$$

$$\tan E' = \frac{Z}{R}\frac{a}{b}$$

$$= (.99665105)\ (1.0033,6361)$$

$$= 1.00000340$$

$$E' = 45° \ 0' \ 0''.35$$

$$\sin E' = .70710798$$

$$\cos E' = .70710558$$

$$-R/a \sec E' = 1.00047872$$

$$+ \frac{Z}{a}\frac{b}{a} \csc E' = \underline{.99378209}$$

$$e^2 = -.00669663$$

$$e^2 = \underline{.00669342}$$

$$\Delta = -.00000321$$

$$R/a \sec E' \tan E' = 1.00048212$$

$$\frac{Z}{a}\frac{b}{a} \csc E' \cot E' = \underline{.99377871}$$

$$1.99426083$$

$$\Delta E = \frac{\Delta}{\sum} = -.00000161$$

$$\Delta E' = -0''.33$$

$$E = E' + \Delta E'$$

$$= 45° \; 0' \; 0''.35 - 0''.33$$

$$= 45° \; 0' \; 0''.02$$

The true value of E is 45° 0' 0".0. Thus, the true value is reached to the significant decimal place on the first iteration In any case

$$\tan \phi = \tan E \frac{a}{b} = \frac{\tan E}{(1 - e^2)^{\frac{1}{2}}}$$

$$X_N = a \cos E \cos \lambda$$

$$Y_N = a \cos E \sin \lambda$$

$$Z_N = b \sin E$$

and

$$\cos \lambda = \frac{X}{R} = .50000003$$

$$\sin \lambda = \frac{Y}{R} = .86602541$$

		Error
ϕ	= 45° 5' 46.334	0".02
λ	= 60°	0".0
X_N	= 7,398,3480	0.0 ft
Y_N	=12,814,314.0	-1.0 ft
Z_N	=14,747,094.0	-1.0 ft

N denotes the geocentric coordinates of the point at sea level.

By the direct method

$$\tan \phi' = \frac{Z}{R} \frac{a^2}{b^2}$$

$$= 1.00336701$$

$$\phi' = 45° \; 5' \; 46''.66 \text{ (error } 0''.31)$$

$$\sin \phi' = .70829419$$

$$\cos \phi' = .70591738$$

$$- R \sec \phi' (1 - e^2) = 20,830,578.2$$

$$\underline{- he^2 = 66.93422}$$

$$- [R \sec \phi' (1 - e^2) + he] = -20,830,645.13442$$

$$\underline{+ Z \csc \phi' = 20,830,578.8}$$

$$= -66.33422$$

$$R \sec \phi' (1 - e^2) \tan \phi' = 20,900,782.12$$

$$\underline{+ Z \csc \phi' \cot \phi' = 20,760,677.39}$$

$$\sum = 41,661,459.51$$

$$\Delta\phi = \frac{\Delta}{\sum} = .00000159$$

$$\Delta\phi = -0''.33$$

$$\phi = \phi' + \Delta\phi''$$

$$= 45° \; 5' \; 46''.33 \text{ (error } 0''.02)$$

Thus either method direct or indirect converges with equal efficiency. The longitude reduction is the same in either case. A check on the reduction is obtained with

$$h = [(X_N - X_N)^2 + (Y - Y_N)^2 + (Z - Z_N)^2]^{\frac{1}{2}}$$

$= 10,001.2$ which is an error of 1.2 feet.

Under the geometric conditions of the model the expanded ellipsoid is a simple method yielding accurate results. The relevant geometry of the model is h = 10,000 feet, $H_o = 528,000.0$ and ζ_o is 61°.

ζ_o is the angle enclosed at the camera station between the camera radius vector and the line in space. Obviously as $\zeta_o \to 0$ errors arising from relief displacement approach zero. Conversely as $\zeta_o \to 90°$ the relief displacement approaches ∞.

4.2 Method Based Zero Change in M, X, Y, Z and ϕ

The second method is illustrated in Figure 7. ζ is the angle enclosed between M and P.

Initially M is computed as if h were zero using a and b in the quadratic in $M = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$

where

$$A = [(\cos^2\alpha + \cos^2\beta)\ (1 - e^2) + \cos^2\gamma]$$
$$B = -2\ [(\cos \alpha X_o + \cos \beta Y_o)\ (1 - e^2) + \cos \gamma Z_o]$$
$$C = [(X_o^{\ 2} + Y_o^{\ 2})\ (1 - e^2) + Z_o^{\ 2} - a(1 - e^2)]$$

Having M to the earth surface ρ. is the radius vector of this part approximated:

$$X = X_o - M \cos \alpha$$
$$Y = Y_o - M \cos \beta$$
$$Z = Z_o - M \cos \gamma$$

$$\tan \phi' = \frac{Z}{(X^2 + Y^2)^{\frac{1}{2}}\ (1 - e^2)}$$

$$\Delta R = h \sin \phi'$$
$$\Delta R = h \cos \phi'$$
$$Z' = Z + \Delta Z$$

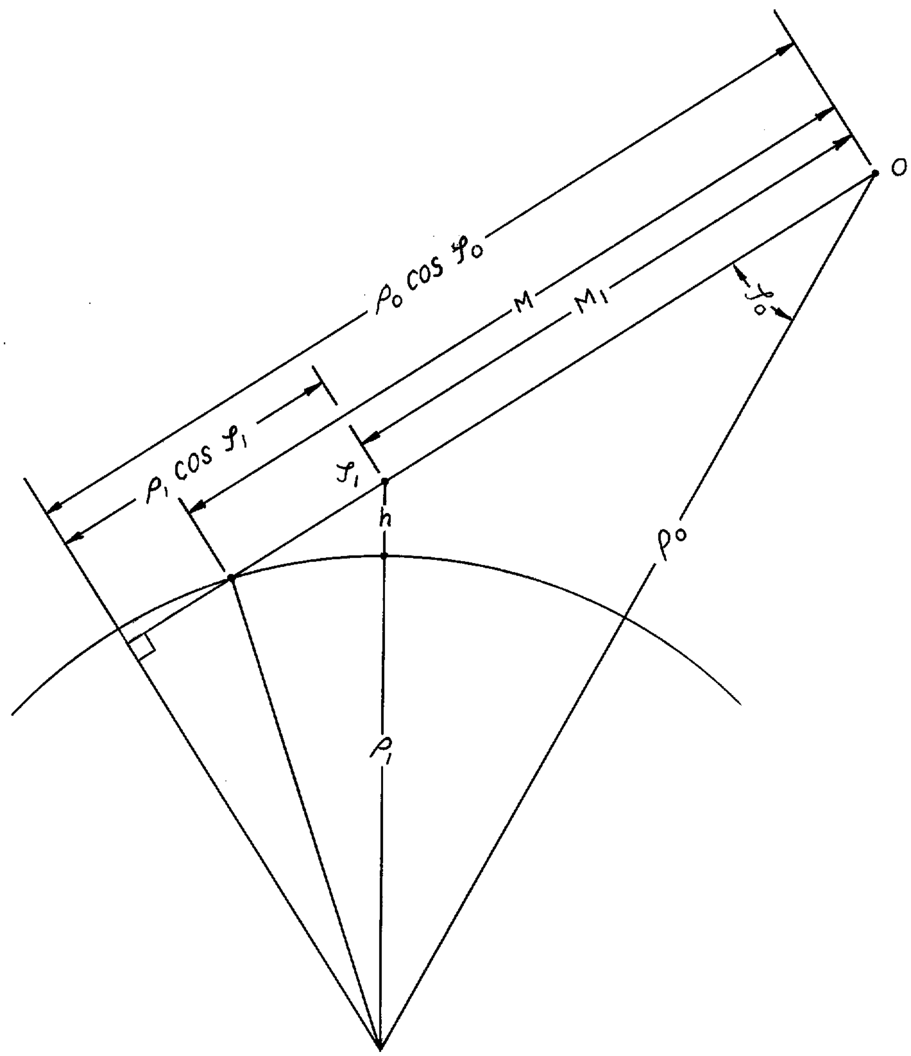


Figure 7 METHOD BASED ON ZERO CHANGE IN M, X, Y, Z, AND Ø

$$R' = R + \Delta R_1$$

$$\rho_1 = (Z'^2 + R'^2)$$

$$\sin \zeta_1 = \frac{\rho_o \sin \zeta_o}{\rho_1}$$

$$M_1 = \rho_o \cos \zeta - \rho_1 \cos \zeta_1$$

Then approximate values of the geocentric coordinates X, Y, Z and latitude (ϕ) are computed:

$$X_1 = X_o - M_1 \cos \alpha$$

$$Y_1 = Y_o - M_1 \cos \beta$$

$$Z_1 = Z_o - M_1 \cos \gamma$$

$$\tan \phi'_1 = \frac{Z_1}{R_1(1 - e^2)}$$

or

$$\tan E'_1 = \frac{Z_1}{R_1}\frac{a}{b} = \frac{Z_1}{R_1 (1 - e^2)^{\frac{1}{2}}}$$

and as before

$$\Delta E = \frac{e^2 - (\frac{R_1}{a} \sec E'_1 - \frac{Z_1}{a}\frac{b}{a} \csc E'_1)}{\frac{R}{a} \sec E'_1 \tan E'_1 + \frac{Z_1}{a}\frac{b}{a} \csc E'_1 \text{ cat } E'_1}$$

$$E_1 = E'_1 + \sum \Delta E$$

Then as in the first iteration

$$\tan \phi = \frac{\tan E}{(1 - e^2)^{\frac{1}{2}}}$$

$$R'_1 = a \cos E_1 + h \cos \phi.$$

On the 2nd iteration

$$X_1 = X_o - M_1 \cos\alpha = 7,401,967.0$$
$$_1 = Y_o - M_1 \cos\beta = 12,820,475.0$$
$$Z_1 = Z_o - M_1 \cos\gamma = 14,754,161.0$$

$$R_1 = (X_1 \quad + \quad Y_1 \;) = 14,803,840.5$$
$$\tan E_1' = \frac{Z_2}{R_2(1 - e^2)^{\frac{1}{2}}}$$
$$= .99999647$$
$$E_1' = 44° \; 59' \; 59''.64$$
$$\sin E_1' = .70710674$$
$$\cos E_1' = .70710801$$

$$e^2 - [\frac{R_1}{a} \sec E_1' - \frac{Z_1}{a} = \frac{b}{a} \csc E_1'] = \Delta = .00000356$$

$$\frac{R_1}{a} \sec E_1' \tan E_1' + \frac{Z_1}{a} = \frac{b}{a} \csc E_1' \cot E_1' = \sum = 1.99426494$$

$$\Delta E = \frac{\Delta}{\sum} = 0.00000178$$

$$\Delta E'' = 0.37$$

$$E_1 = E_1' + \Delta E''$$

$$= 45° \; 0' \; 0''.01$$

$$\sin E_1 = .70710635$$
$$\cos E_1 = .70710675$$

$$\tan \phi_1 = \tan E \;= 1.00336370$$
$$\phi_1 = (1 - e^2)^{\frac{1}{2}} = 45° \; 5' \; 46''.32$$

$$R_1' = a \cos E_1 + h \cos \phi_1 = 14,803,755.0$$
$$Z_1' = b \sin E_1 + h \sin \phi_1 = 14,754,177.0$$

$$\rho_2 = (Z_1'^2 + R_1'^2)^{\frac{1}{2}} = 20,900,644.0 \quad \rho_1 - \rho_2 = 49 \text{ ft.}$$

$a = 20{,}925{,}689.0 \qquad a(1 - e^2) = 20{,}785{,}625.0$

$e^2 = .00669342$

$(1 - e^2 \sin^2\phi_1) = .99663737$

$(1 - e^2 \sin^2\phi_1)^{1/2} = .99831727$

$(1 - e^2 \sin^2\phi_1)3/2 = .99496030$

$m = 20{,}900{,}909.0$

$n = 20{,}970{,}961.0$

$\sin\phi_1 \cos\lambda_1 = 0.35394455 \qquad \cos\phi_1 \sin\lambda_1 = 0.61117156$

$\sin\phi_1 \sin\lambda_1 = 0.61408666 \qquad \cos\phi_1 \cos\lambda_1 = 0.35226435$

$\cos\phi_1 = 0.70542246$

Condition Equations

$\cos\alpha\cdot\Delta M$	$A\cdot\Delta\phi$	$+ B\cdot\Delta\lambda$	$= dX$
0.87122760	-7,397,763.0	-12,816,855.0	= -3523.0
$\cos\beta\cdot\Delta M$	$A_2\cdot\Delta\phi$	$B_2\cdot\Delta\lambda$	$= dY$
0.46326245	-12,834,969.0	7,387,322.0	= -6112.0
$\cos\gamma\cdot\Delta M$	$A_3\cdot\Delta\phi$	$B_3\cdot\Delta\lambda$	$= dZ$
-0.16233007	14,743,971.0	0.0	= -7088.0

Let $dM = X\cdot 10^{+3}$

$\Delta\phi = Y\cdot 10^{-4}$

$\Delta\lambda = Z\cdot 10^{-4}$

$$Z_1' = b \sin E_1 + h \sin \phi.$$

$$\rho_2 = (R_1'^2 + Z_1'^2)^{1/2}$$

$$\sin \zeta_2 = \frac{\rho_o \sin \zeta_o}{2}$$

$$M_2 = \rho_o \cos \zeta_o - \rho_2 \cos \zeta_2$$

When two successive values of M, X, Y, Z and ϕ are obtained without signficant numerical change final values ϕ and λ are computed with final values of X, Y, Z which are computed with the final value of M.

4.2.1 Numerical Example

Using the data from the previous model

$$a = 20{,}925{,}689.0$$
$$b = 20{,}855{,}539.0$$
$$e = .006693422$$
$$(1-e^2) = .99330658$$
$$a\,(1-e^2) = 20{,}785{,}625.0$$

$$X_o = 8{,}314{,}238.0$$
$$Y_o = 13{,}305{,}562.0$$
$$Z_o = 14{,}584{,}184.0$$
$$\cos \alpha = .87122760$$
$$\cos \beta = .46326245$$
$$\cos \gamma = -.16233007$$

$$A = [(\cos^2 \alpha + \cos^2\beta)\ (1 - e^2) + \cos^2\gamma]$$
$$= .99348364$$
$$B = -2[(\cos \alpha\ X_o + \cos \beta\ Y_o)\ (1 - e^2) + \cos \gamma\ Z_o]$$
$$= -21{,}900{,}734.0$$
$$C = [X_o^2 + Y_o^2)\ (1 - e^2) + Z_o^2 - a^2(1 - e^2)]$$
$$= 22.261{,}74961 \times 10^{12}$$

$M = 1{,}067{,}893.95$

$M' = 20{,}973{,}138.51$

The shorter value is the desired value.

$X = X_o - M \cos \alpha = 7{,}383{,}859.0$

$Y = Y_o - M \cos \beta = 12{,}810{,}847.0$

$Z = Z_o - M \cos \gamma = 14{,}757{,}535.0$

$R = (X^2 + Y^2)^{1/2} = 14{,}786{,}452.4$

$$\tan \phi' = \frac{Z}{R(1-e^2)}$$

$= 1.00476968$

$\phi' = 45° \; 8' \; 10''.74$

$\Delta Z = h \sin \phi' = 7087.87$

$\Delta R = h \cos \phi' = 7054.22$

$R + \Delta R = R_1' = 14{,}793{,}507.6$

$Z + \Delta Z = Z_1' = 14{,}764{,}623.0$

$\rho_1 = (R'^2 + Z'^2)^{1/2} = 20{,}900{,}693.0$

$\rho_o = (X_o^2 + Y_o^2 + Z_o^2)^{1/2}$

$= 21{,}421{,}086.73$

$$\cos \zeta_o = \cos \alpha \frac{X_o}{\rho_o} + \cos \beta \frac{Y_o}{\rho_o} + \cos \gamma \frac{Z_o}{\rho_o}$$

$= .51538511$

$\zeta_o = 58° \; 58' \; 36''.43$

$\sin \zeta_o = .85695868$

$$\sin \zeta_1 = \frac{\rho_o \sin \zeta_1}{\rho}$$

$= .8782{,}9558$

$\zeta = 61° \; 26' \; 14''.78$

$\cos\zeta_1 = .47811806$

$\rho_o \cos \zeta_o = 11{,}040{,}109.0$

$\rho_1 \cos \zeta_1 = 9{,}992{,}999.0$

$M_1 \quad 1{,}047{,}110.0$

$$\sin \zeta_2 = \frac{\rho_o \sin \zeta_o}{\rho}$$

$$= .87829764$$

$$\zeta_1 = 61° \ 26' \ 15''.67 \qquad \zeta_1 - \zeta_2 = 0.89$$

$$\cos \zeta_2 = .47811427$$

$$\rho_o \cos \zeta_o = 11.040,109.0$$

$$\rho_2 \cos \zeta_2 = 9,992,896.0$$

$$M_2 = 1,047,213.0 \qquad M$$

$$X_2 = X_o - M_2 \cos \alpha = 7,401,877 \qquad X_1 - X_2 = 90$$

$$Y_2 = Y_o - M_2 \cos \beta = 12,820,428 \qquad Y_1 - Y_2 = 47$$

$$Z_2 = Z_o - M_2 \cos \gamma = 14,754,178 \qquad Z_1 - Z_2 = -17$$

$$R_2 = (X_2{}^2 + Y_2{}^2)^{\frac{1}{2}}$$

$$= 14,803,755.0 \qquad R_1 - R_2 = 95.0$$

$$\tan E_2' = \frac{Z_2}{R_2 \ (1 - e^2)^{\frac{1}{2}}}$$

$$= 1.00000341$$

$$E'_2 = 45° \ 0' \ 0''.35$$

$$\cos E_2' = .70710558$$

$$\sin E_2' = .70710798$$

$$e^2 - [\frac{R_2}{a} \sec E_2' - \frac{Z_2}{a} = \frac{b}{a} \csc E_2'] = \Delta = -.00000320$$

$$\frac{R_2}{a} \sec E_2' \tan E_2' + \frac{Z_3}{a} \frac{b}{a} \csc E_2' \cot E_2' = \sum = 1.99426082$$

$$\Delta E_2 - \frac{\Delta}{\sum} = .00000160$$

$$\Delta E'' = -0.33$$

$$E_2 = E_2' + \Delta E'' =$$

45° 0' 0".02

$$\tan \phi_2 = \frac{\tan E_2}{(1 - e^2)^{\frac{1}{2}}} = 1.00336371$$

R $\qquad \phi_2 = 45° \; 5' \; 46''.36 \qquad\qquad \phi_2 - \phi_1 = 0''.04$

$R_2' = a \cos E_3 + h \cos \phi_3 = 14,803,754 \qquad\qquad R_2 - R_1 = 1.0$

$Z_2' = b \sin E_3 + h \sin \phi_3 = 14,754,177 \qquad\qquad Z_2 - Z_1 = 1.0$

Since the change in ϕ, R, and Z is negligible, the change in ρ, M, and ζ must also be negligible.

Therefore, the final values are

$$\phi_2 = 45° \; 5' \; 46''.36$$

$$\cos \lambda_2 = \frac{X_2}{R_2} = .499,999\ 98$$

$$\sin \lambda_2 = \frac{Y_2}{R_2} = .86602544$$

$$\lambda_2 = 60°$$

Latitude converges within 0".04 and the longitude to within 0".0.

4.3 Method Based on Iteration of M, ϕ, and λ

In this method M is computed as if h were zero followed by adjusting the geocentric coordinates for elevation of the point. The geocentric coordinates corrected for h combined with M cos α, M cos β, M cos γ yield X'_o, Y'_o, and Z'_o whose errors are a function of the errors in M, ϕ, and λ.

Initially ϕ and γ are approximated.

$$M = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$X = X_o - M_1 \cos \alpha$$

$$Y = Y_o - M_1 \cos \beta$$

$$Z = Z_o - M_1 \cos \gamma$$

$$\tan \phi_1 = \frac{Z}{R\,(1 - e^2)^{\frac{1}{2}}}$$

$$\cos \lambda_1 = \frac{X}{R}$$

$$\sin \lambda_1 = \frac{Y}{R}$$

Introducing h the geocentric coordinates are adjusted

$$X_1 = \left(\frac{a}{\sqrt{1 - e^2 \sin^2 \phi_1}} + h \right) \cos \phi \quad \cos \lambda_1$$

$$Y_1 = \left(\frac{a}{\sqrt{1 - e^2 \sin^2 \phi_1}} + h \right) \cos \phi \quad \sin \lambda_1$$

$$Z_1 = \left(\frac{a\,(1 - e\,)}{\sqrt{1 - e^2 \sin \phi_1}} + h \right) \sin \phi_1$$

Then

$$dX_1 = X_o - (M \cos\alpha + X_1)$$

$$dY_1 = Y_o - (M \cos\beta + Y_1)$$

$$dZ_1 = Z_o - (M \cos\gamma + Z_1)$$

The above residuals are functions of erros in M, ϕ, and λ. Differentiating M, ϕ, and λ with respect to dX_o, dY_o, and dZ_o.

$$dX = -\left[\frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)3/2} + h\right] \sin\phi \cos\lambda \, \Delta\phi - \left[\frac{a}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}} + h\right] \cos\phi \sin\lambda \, \Delta\lambda$$

$$dY = -\left[\frac{a(1 - e^2)}{(1 - e^2 \sin^2\phi)3/2} + h\right] \sin\phi \sin\lambda \, \Delta\phi + \left[\frac{a}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}} + h\right] \cos\phi \cos\lambda \, \Delta\lambda$$

$$dZ = \left[\frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)3/2} + h\right] \cos\phi \, \Delta\phi$$

or

$$\cos\alpha\cdot\Delta M + A_1\Delta\phi + B_1\Delta\lambda = dX$$
$$\cos\beta\cdot\Delta M + A_2\Delta\phi + B_2\Delta\lambda = dY$$
$$\cos\gamma\cdot\Delta M + A_3\Delta\phi + B_3\Delta\lambda = dZ$$

where

$$A_1 = -\left[\frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)3/2} + h\right] \sin\phi \cos\lambda$$

$$B = -\left[\frac{a}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}} + h\right] \cos\phi \sin\lambda$$

$$A_2 = -\left[\frac{a\ (1 - e^2)}{(1 - e^2\ \sin^2\ \phi)3/2} + h\right] \sin\phi\ \sin\lambda$$

$$B_2 = \left[\frac{a}{(1 - e^2\ \sin^2\ \phi)^{1/2}} + h\right] \cos\phi\ \cos\lambda$$

$$A_3 = \left[\frac{a(1 - e^2)}{(1 - e^2\ \sin^2\ \phi)3/2} + h\right] \cos\phi$$

$$B_2 = 0$$

Solving the above 3 x 3 for ΔM, $\Delta\phi$, $\Delta\lambda$

$$M_2 = M_1 + dM_1$$

$$\phi_2 = \phi_1 + \Delta\phi_1$$

$$\lambda_2 = \lambda_1 + \Delta\lambda_1$$

Whence

$$X_2 = \left[\frac{a}{(1 - e^2\ \sin^2\phi_2)^{1/2}} + h\right] \cos\phi_2\ \cos\lambda_2$$

$$Y_2 = \left[\frac{a}{(1 - e\ \ \sin^2\phi_2)^{1/2}} + h\right] \cos\phi_2\ \sin\lambda_2$$

$$Z_2 = \left[\frac{a(1 - e^2)}{(1 - e^2\ \sin^2\phi_2)^{1/2}} + h\right] \sin\phi_2$$

Then again

$$dX_2 = X_o - (M_2 \cos\alpha + X_2)$$

$$dY_2 = Y_o - (M_2 \cos\beta + Y_2)$$

$$dZ_2 = Z_o - (M_2 \cos\gamma + Z_o)$$

again the 3 x 3 is formed and solved for dM_2, $\Delta\phi_2$, and $\Delta\gamma_2$. This process is repeated until

$$dX_n = 0$$

$$dY_n = 0$$

$$dZ_n = 0$$

and

$$M_n = M_{n-1} + dM_{n-1}$$

$$\phi_n = \phi_{n-1} + \Delta\phi_{n-1}$$

$$\lambda_n = \lambda_{n-1} + \Delta\lambda_{n-1}$$

are the final values where n is the iteration in which the residuals vanish.

4.3.1 Numerical Example

Given X_o, Y_o, Z_o and cos α, cos β, cos γ

M is determined as if h were zero

$$M = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= 1,067,894.0$$

$$X = X_o - M \cos \alpha = 7,383,859.0$$
$$Y = Y_o - M \cos \beta = 12,810,847.0$$
$$Z = Z_o - M \cos \gamma = 14,757,535.0$$
$$R = (X^2 + Y^2)^{1/2} = 14,786,452.0$$

Approximate latitude and longitude

$$\tan \phi_1 = \frac{Z}{R(1 - e^2)^{1/2}} = 1.00476968$$

$\phi_1 = 45° \; 8' \; 10''.74$

$\sin \phi_1 = .70878711$

$\cos \phi_1 = .70542246$

$\tan \lambda_1 = \frac{Y}{X} = 1.73497990$

The coordinates of the surface point are adjusted for h = 10,000 ft.

$X_1 = X + h \cos \phi_1 \cos \lambda_1 = 7,387,382.0$

$Y_1 = Y + h \cos \phi_1 \sin \lambda_1 = 12,816,959.0$

$Z_1 = Z + h \sin \phi_1 = 14,764,623.0$

$dX = X_o - (M \cos \alpha + X_1) = -3523.0$

$dY = Y_o - (M \cos \beta + Y_1) = -6112.0$

$dZ = Z_o - (M \cos \gamma + Z_1) = -7088.0$

Let

$$m = \frac{a(1 - e^2)}{(1 - e^2 \sin^2\phi_1)^{3/2}} + h$$

$$n = \frac{a}{(1 - e^2 \sin^2 \phi_1)^{1/2}} + h$$

$A_1 = -m \sin \phi \cos \lambda \qquad B_1 = -n \cos \phi \sin \lambda$

$A_2 = -m \sin \phi \sin \lambda \qquad B_2 = n \cos \phi \cos \lambda$

$A_3 = m \cos \phi \qquad B_3 = 0$

$\sin \lambda_1 = .8663,9084$

$\cos \lambda_1 = .4993,6650$

Whence

X	Y	Z	=	r
871.22760	-739.77630	-1,281.6855	=	-3523.0
463.26245	-1,283.49690	738.7322	=	-6112.0
-162.33007	1,474.39710	0.0	=	-7088.0

Eliminating Z from the first two equations by successive division and subtraction:

(1)	0.67975147	-0.57719019	-1.0	= -2.74872424
(2)	0.62710472	-1.73743191	-1.0	= -8.27363420
(1)&(2)	1.30685619	-2.31462210	0.0	=-11.02235840
(3)	-162.33007	1,474.3971	0.0	=-7088.0

Eliminating X from the resultant of (1), (2), and (3):

(4)	1.0	-1.77113757	=-8.43425506
(5)	-1.0	9.08271092	=-43.66412211
		7.31157335	=-52.09837717

Y = -7.125,467,34

$\Delta\phi = Y \times 10^{-4} = -.000712550$

$\Delta\phi'' = \dfrac{\Delta\phi}{\sin 1''}$

$\sin 1'' = .00000484813672$

$\Delta\phi'' = -146''.97$

$= -2' \ 26''.97$

Substituting in (4) and (5) to obtain X in the back solution:

	- 8.43425506	-43.66412211
	-12.62018291	64.71856002
X =	-21.05443797	+21.05443791

$$dM = 1000X$$
$$= -21{,}054.44 \text{ ft.}$$

Substituting X and Y in (1) and (2) to obtain Z in the back solution:

	- 2.74872424	- 8.27363420
	- 4.11274985	-12.38001433
	- 6.86147409	-20.65364853
	14.31178514	13.20333741
Z =	7.45031105	- 7.45031112

$$\Delta\lambda = Z\ 10^{-4}$$
$$= -.0007450311$$
$$\Delta\lambda'' = \Delta\lambda/\sin 1''$$
$$= -153''.67$$
$$= -2'\ 33''.67$$

These corrections are applied to the first approximations and the condition equations are reformed:

M_1	1,067,894	ϕ_1	45°	8'	10''.74	λ_1	60°	2'	30''.85
ΔM	- 21,054	-		2'	26''.97	-		2'	33''.67
M_2	1,046,840	ϕ_2	45°	5'	43''.77	λ_2	59°	59'	57''.18

In the 2nd iteration

$$dX_2 = X_o - [M_2 \cos\alpha + \left(\frac{a}{\sqrt{1 - e^2 \sin^2\phi_2}} + h\right) \cos\phi_2 \cos\lambda_2]$$

$$dY_2 = Y_o - [M_2 \cos\beta + \left(\frac{a}{\sqrt{1 - e^2 \sin^2\phi_2}} + h\right) \cos\phi_2 \sin\lambda_2]$$

$$dZ_2 = Z_o - [M_2 \cos\gamma + \left(\frac{a(1-e^2)}{\sqrt{1-e^2\sin^2\phi_2}} + h\right)\sin\phi_2]$$

	ϕ_2	λ_2
cos	.70592730	.50001184
sin	.70828429	.86601857

$$\cos\phi_2 \cos\lambda_2 = 0.35297201$$

$$\cos\phi_2 \sin\lambda_2 = 0.61134615$$

$$1 - e^2\sin^2\phi_2 = .99664213$$

$$\sqrt{1 - e^2\sin^2\phi_2} = .99831965$$

$$\frac{a}{\sqrt{1-e^2\sin^2\phi_2}} + h = 20{,}970{,}911.0$$

$$\frac{a(1-e^2)}{\sqrt{1-e^2\sin^2\phi_2}} + h = 20{,}830{,}611.0$$

$$X_o - [M_2\cos\alpha + \left(\frac{a}{\sqrt{1-e^2\sin^2\phi_2}} + h\right)\cos\phi_2\cos\lambda_2] = dX$$

$$= 8{,}314{,}238 - [912{,}036.0 + 7{,}402{,}145] = 57.0$$

$$Y_o - [M_2\cos\beta + \left(\frac{a}{\sqrt{1-e^2\sin^2\phi_2}} + h\right)\cos\phi_2\sin\lambda_2] = dY$$

$$13{,}305.562 - [484{,}962 + 12{,}820{,}486] = 114.0$$

$$Z_o - [M_2\cos\lambda + \left(\frac{a(1-e^2)}{\sqrt{1-e^2\sin^2\phi_2}} + h\right)\sin\phi_2] = dZ$$

$$14{,}584{,}184 - [-169{,}934 + 14{,}753{,}995] = 123.0$$

$$m_2 = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi_2)^{3/2}} + h = 20{,}900{,}759.0$$

$$n_2 = \frac{a}{(1 - e^2 \sin^2 \phi_2)} + h = 20{,}970{,}911.0$$

$\sin \phi_2 \cos \lambda_2 =$ 0.3541,5053

$\sin \phi_2 \sin \lambda_2 =$ 0.6133,8735

Condition Equations

	X	Y	Z		r
(1)	871.22760	-740.201488	-1282.048510	=	57.0
(2)	463.26245	-1282.026118	740.214461	=	114.0
(3)	-162.33007	1480.367925	0.0	=	123.0

Solving by successive division and subtraction in the order of previous reduction.

(1)	0.67955897	- .57735838	- 1.0	= 0.04446009
(2)	0.62584896	- 1.73196578	1.0	= 0.15400942
(1) & (2)	1.30540793	- 2.30932416	0.0	= 0.19846951
(3)	-162.233007	1480.367925	0.0	= 123.0
(4)	1.0	- 1.769044071		= 0.15203639
(5)	-1.0	9.12494906		= 0.75816877
		7.35590499		= 0.91020516

$Y =$ 0.1237380256

$\Delta\phi = Y \cdot 10^{-4} =$ 0.0000123738

$\Delta\phi'' = \Delta\phi / \sin 1'' = + 02''.55$

Back solution for X

	0.15203639	0.75816877
	0.21889802	-1.12910310
X	0.37093441	-0.37093441

$$dM = X \cdot 10^{+3} = 370.9 \text{ ft.}$$

Back Solution for Z

	.04446009	0.15400942
	.07144119	0.21431003
	0.11590128	0.36831945
	- .25207181	- .23214891
Z =	- .13617053	.13617054

$$= Z \cdot 10^{-4} = .000013617054$$

$$\Delta\lambda'' = \Delta\lambda / \sin 1'' = 02''.81$$

Summary of Iterations

	M ft.	ϕ	λ
1st Approximation	1,067,894	45° 8' 10".74	60° 2' 30".85
1st Correction	- 21,054	- 2' 26".97	- 2' 33".67
2nd Approximation	1,046,840	45° 5' 43".77	59° 59' 57".18
2nd Correction	371	+ 2".55	+ 2".81
3rd Approximation	1,047,211	45° 5' 46".32	59° 59' 59.99
True Values	1,047,212	45° 5' 46".31	60°
Error	- 1.0 ft.	+ 0".01	- 0".01

The advantage of the third method is the fact that it converges to the true value without the tedious iteration for

$$\Delta E = \frac{e^2 - \left(\frac{R}{a} \sec E' - \frac{Z}{a}\frac{b}{a} \csc E'\right)}{\frac{R}{a} \sec E' \tan E' + \frac{Z}{a}\frac{b}{a} \csc E' \cot E'}$$

or

$$\Delta\phi = \frac{-\left[h_e^{\,2} + R \sec \phi'(1-e^2) - Z \csc \phi'\right]}{R \sec \phi'(1-e^2) \tan \phi' + Z \csc \phi' \cot \phi'}$$

5. NUMERICAL EVALUATION

The numerical examples indicate the first method is the most expeditious and the third method is the most rigorous and general. Conceivably the expanded ellipsoid method might yield inaccurate position coordinates for abnormally high altitudes. The expanded ellipsoid method should be tested through the range of altitudes found in the earth surface features. A range of h values from 1000 ft. to 30,000 ft. in increasing increments of 1000 ft. would cover this geometric aspect of the test. Earth surface detail has little or no interpretation value as specific points for camera stations higher than 500 miles. Therefore testing the first and third methods on earth surface points ranging 1000 ft. to 30,000 ft. at camera stations ranging in altitude from 100 miles to 500 miles would provide numerical data on all possible geometries that the methods might be applied to in practice. A range of 30,000 ft. for h in 1000 ft. increments combined with a range of 500 miles for H_o in 100 mile increments requires the reduction of each of the two methods to 5 x 30 models or 300 reductions. Geocentric coordinates of the camera station and the earth surface point and the direction cosines of the corresponding lines in space must be generated for each model. The latitude (ϕ), longitude (λ) and parametric angle (E) are fixed for both the camera station and earth surface point. The geocentric coordinates of the camera station (X_o, Y_o, Z_o) vary with H; the geocentric coordinates of the surface point (X, Y, X) vary with h; and the direction cosines of the corresponding lines vary with either a change in h or H.

Assume the following constant data.

Earth Surface Point	Camera Station
$E = 45°$ N	$E_o = 43°$ N
$\lambda = 60°$ W	$\lambda_o = 58°$ W

$$a = 20{,}925{,}689 \text{ feet}$$
$$b = 20{,}855{,}539 \text{ feet}$$
$$\tan \phi = \tan E \frac{a}{b}$$

$$\tan \phi = 1.00336361 \qquad \tan \phi_o = 0.93565171$$
$$\phi = 45° \; 5' \; 46''.3 \qquad \phi_o = 43° \; 5' \; 45''.5$$

The constant geocentric coordinates are those at the datum surface.

$$X_k = a \cos E \cos \lambda \qquad X_{ok} = a \cos E_o \cos \lambda_o$$
$$Y_k = a \cos E \sin \lambda \qquad Y_{ok} = a \cos E_o \sin \lambda_o$$
$$Z_k = b \sin E \qquad A_{ok} = b \sin E_o$$

	E	λ	E_o	λ_o
cos	.70710678	.500000	.73135370	.52991926
sin	.70710678	.86602540	.68199836	.84804810

$$X_k = 7{,}398{,}348.0 \qquad X_{ok} = 8{,}109{,}927.0$$
$$Y_k = 12{,}814{,}315.0 \qquad Y_{ok} = 12{,}978{,}596.0$$
$$Z_k = 14{,}747{,}095.0 \qquad Z_{ok} = 14{,}223{,}443.0$$

The geocentric coordinates of the earth surface point and camera station with any value of h and H are simply:

$$X_1 = X_k + h \cos \phi \cos \lambda \qquad X_{o1} = X_{ok} + H \cos \phi_o \cos \lambda_o$$
$$Y_1 = Y_k + h \cos \phi \sin \lambda \qquad Y_{o1} = Y_{ok} + H \cos \phi_o \sin \lambda_o$$
$$Z_1 = Z_k + h \sin \phi \qquad Z_{o1} = Z_{ok} + H \sin \phi_o$$

The coefficients of h, $\cos \phi \cos \lambda$, $\cos \phi \sin \lambda$ and $\sin \phi$ are constant for all values of h just as the coefficient of H, $\cos \phi_o \cos \lambda_o$, $\cos \phi_o \sin \lambda_o$ and $\sin \phi_o$ are constant for all values of H.

	ϕ	ϕ_o
cos	.70591861	.73021028
sin	.70829296	.68322244

To provide a numerical example

$$h = 10{,}000 \text{ ft.} \quad \text{and} \quad H = 528{,}000 \text{ ft.}$$

are reduced.

$$\begin{aligned} h \cos\phi \cos\lambda &= 3{,}530.0 & H \cos\phi_o \cos\lambda_o &= 204{,}311.0 \\ h \cos\phi \sin\lambda &= 6{,}113.0 & H \cos\phi_o \sin\lambda_o &= 326{,}966.0 \\ h \sin\phi &= 7{,}083.0 & H \sin\phi_o &= 360{,}741.0 \end{aligned}$$

whence

$$\begin{aligned} X_1 &= X_k + h \cos\phi \cos\lambda = 7{,}401{,}878.0 \\ Y_1 &= Y_k + h \cos\phi \sin\lambda = 12{,}820{,}428.0 \\ Z_1 &= Z_k + h \sin\phi_o = 14{,}754{,}178.0 \end{aligned}$$

These are the quantities to be subsequently determined.

$$\begin{aligned} X_o &= X_{ok} + H \cos\phi_o \cos\lambda_o = 8{,}314{,}238.0 \\ Y_o &= Y_{ok} + H \cos\phi_o \sin\lambda_o = 13{,}305{,}562.0 \\ Z_o &= Z_{ok} + H \sin\phi_o = 14{,}584{,}184.0 \end{aligned}$$

While X_1, Y_1, Z_1 are subsequently solved they are employed in the construction of the model to determine the exact values of cos α, cos β, cos γ and M.

$$\cos\alpha = \frac{X_o - X_1}{M}$$

$$\cos\beta = \frac{Y_o - Y_1}{M}$$

$$\cos\gamma = \frac{Z_o - Z_1}{M}$$

$$M = [(X_o - X_1)^2 + (Y_o - Y_1)^2 + (Z_o - Z_1)^2]^{\frac{1}{2}}$$

$$X_o - X_1 = 912,360.0$$

$$Y_o - Y_1 = 485,134.0$$

$$Z_o - Z_1 = -169,994.0$$

$$M = 1,047,212.0 \text{ ft.}$$

$$\cos \alpha = .87122760$$

$$\cos \beta = .46326245$$

$$\cos \gamma = -.16233007$$

The input to each reduction is X_o, Y_o, Z_o, $\cos \alpha$, $\cos \beta$, $\cos \gamma$ and datum constants a and b. The values to be determined in each case are M, ϕ and λ. Any other model is generated by changing h and H as specified. The reason for selecting E = 45° is to subject the model to any errors arising from the difference in geocentric and geodetic latitude. This difference is illustrated in Figure 8. The geocentric latitude is defined by the line from the center of the earth to the point and the equator. It decreases with h. Geodetic latitude is defined by a perpendicular to an ellipsoid of revolution at the point and the equator. It is independent of elevation. The difference $\phi - \phi_c = \Delta\phi$ is zero at the equator and the pole. It can be shown that the difference is a maximum at $\phi_c = 45°$.

$$\tan \phi = \frac{Z}{R(1-e^2)}$$

$$\tan \phi_c = \frac{Z}{R}$$

$$\therefore \quad \tan \phi = \frac{\tan \phi_c}{(1-e^2)}$$

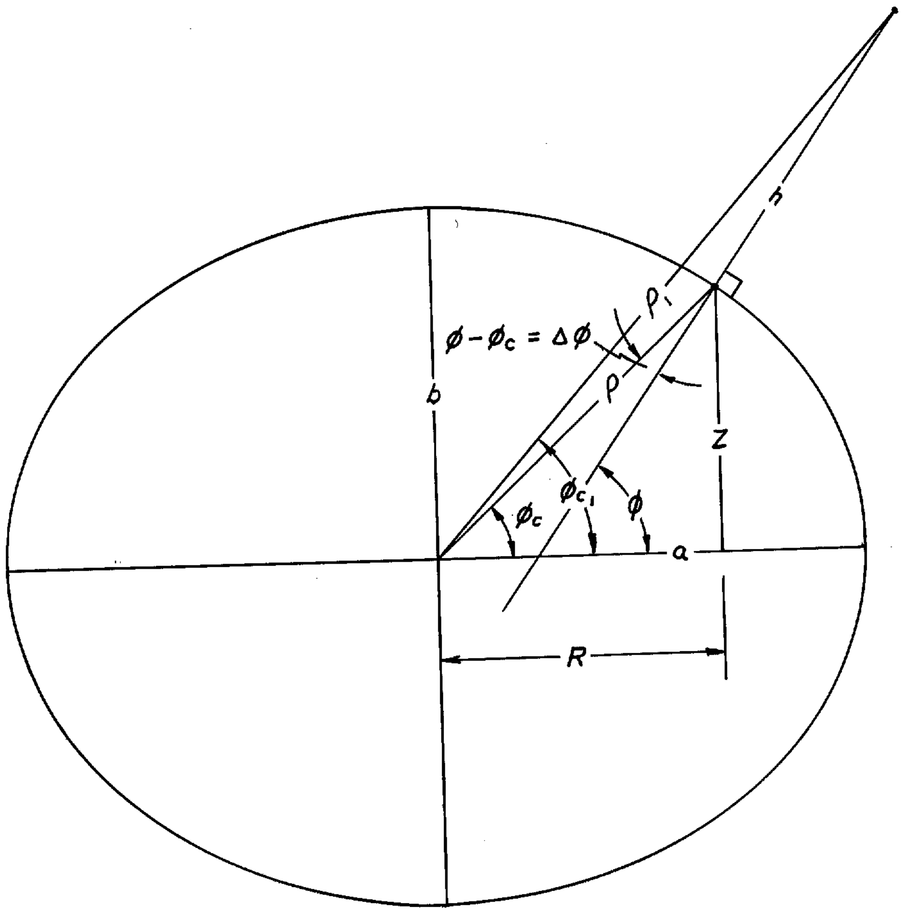


Figure 8 GEOCENTRIC AND GEODETIC LATITUDE

$$\tan \Delta\phi = \frac{\tan \phi - \tan \phi_c}{1 + \tan \phi \tan \phi_c}$$

Substituting

$$= \frac{\dfrac{\tan \phi_c}{(1-e^2)} - \tan \phi_c}{1 + \dfrac{\tan^2\phi_c}{(1-e^2)}}$$

$$= \frac{\tan \phi_c - (1-e^2)\tan \phi_c}{(1-e^2) + \tan^2\phi_c}$$

$$= \frac{e^2\tan \phi_c}{(1-e^2) + \tan^2\phi_c}$$

$$= \frac{e^2\sin \phi_c \cos \phi_c}{1 - e^2\cos^2\phi_c}$$

$$= \frac{e^2\sin 2\phi_c}{2(1-e^2\cos^2\phi_c)}$$

$\sin 2\phi_c$ is a maximum when

$$\phi_c = 45°$$

and when

$$\phi_c = 45°$$

$$\sin 2\phi_c = 1$$

$$\sin \phi_c = \cos \phi_c$$

$$\therefore \quad 1-e^2\cos^2\phi_c = 1 - \frac{e^2}{2} \sin 2\phi_c$$

$$= 1 - \frac{e^2}{2}$$

or tan is max when $\Delta\phi_c$ = 45" and when ϕ_c = 45°

$$\tan \Delta\phi = \frac{e^2}{2 - e^2}$$

$$= \frac{.0066934}{1.9933066}$$

$$= .003357938$$

$$\Delta\phi'' = \frac{\tan \Delta\phi}{\sin 1''}$$

$$= 692''$$

$$= 11' \, 32''$$

Thus the choice E = 45° subjected the methods to the maximum difference between geocentric and geodetic.

or $\tan \Delta\phi$ is maximum when

$$\phi_c = 45°$$

and when

$$\phi_c = 45°$$

$$\tan \Delta\phi = \frac{e^2}{2 - e^2}$$

$$= \frac{.0066934}{1.9933066}$$

$$= .003357938$$

$$\Delta\phi'' = \frac{\tan \Delta\phi}{\sin 1''}$$

$$= 11' \; 32'' \; .$$

Thus, the choice E = 45° subjects the methods to the maximum difference between geocentric and geodetic.